

# Uniform Regularity and Vanishing Viscosity Limit for the Nematic Liquid Crystal Flows in Three Dimensional Domain

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## Abstract

In this paper, we investigate the uniform regularity and vanishing limit for the incompressible nematic liquid crystal flows in three dimensional bounded domain. It is shown that there exists a unique strong solution for the incompressible nematic liquid crystal flows with boundary condition in a finite time interval which is independent of the viscosity. The solution is uniformly bounded in a conormal Sobolev space. Finally, we also study the convergence rate of the viscous solutions to the inviscid ones.

Keywords: nematic liquid crystal flows, vanishing viscosity limit, convergence rate, conormal Sobolev space.

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## 1 Introduction

In this paper, we consider the incompressible nematic liquid crystal flows as follows

$$\begin{cases} u_t^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon = \varepsilon \Delta u^\varepsilon - \nabla d^\varepsilon \cdot \Delta d^\varepsilon, & (x, t) \in \Omega \times (0, T), \\ d_t^\varepsilon + u^\varepsilon \cdot \nabla d^\varepsilon = \Delta d^\varepsilon + |\nabla d^\varepsilon|^2 d^\varepsilon, & (x, t) \in \Omega \times (0, T), \\ \operatorname{div} u^\varepsilon = 0, & (x, t) \in \Omega \times (0, T). \end{cases} \quad (1.1)$$

Here  $0 < T \leq +\infty$  and  $\Omega$  is a bounded smooth domain of  $\mathbb{R}^3$ . The unknown vector functions  $u^\varepsilon(x, t) = (u_1^\varepsilon(x, t), u_2^\varepsilon(x, t), u_3^\varepsilon(x, t))$ ,  $d^\varepsilon(x, t) = (d_1^\varepsilon(x, t), d_2^\varepsilon(x, t), d_3^\varepsilon(x, t))$  and scalar function  $p^\varepsilon(x, t)$  represent the velocity field of fluid, the macroscopic average of the nematic liquid crystal orientation field, and pressure respectively. The parameter  $\varepsilon > 0$  is the inverse of the Reynolds number. Here

$$(u^\varepsilon \cdot \nabla u^\varepsilon)_i = \sum_{j=1}^3 u_j^\varepsilon \partial_j u_i^\varepsilon, \quad (\nabla d^\varepsilon \cdot \Delta d^\varepsilon)_i = \sum_{j,k=1}^3 \partial_i d_j^\varepsilon \partial_{kk} d_j^\varepsilon,$$

for  $i = 1, 2, 3$ . The system (1.1), first proposed by Lin [1], is a simplified version of the general Ericksen-Leslie system modeling the hydrodynamic flow of nematic liquid crystal materials proposed

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by Ericksen [2] and Leslie [3] during the period between 1958 and 1968. System (1.1) is a macroscopic continuum description of the time evolution of the material under the influence of both the fluid velocity field and the macroscopic description of the microscopic orientation configurations of rod-like liquid crystals. The interested readers can refer to [1–3], and Lin-Liu [4] for more details. Corresponding to the system (1.1), we impose the following Navier-slip type and Neumann boundary conditions:

$$u^\varepsilon \cdot n = 0, \quad ((Su^\varepsilon)n)_\tau = -(Au^\varepsilon)_\tau, \quad \text{and} \quad \frac{\partial d^\varepsilon}{\partial n} = 0, \quad \text{on } \partial\Omega, \quad (1.2)$$

where  $A$  is a given smooth symmetric matrix (see [5]),  $n$  is the outward unit vector normal to  $\partial\Omega$ ,  $(Au^\varepsilon)_\tau$  represents the tangential part of  $Au^\varepsilon$ . The strain tensor  $Su^\varepsilon$  is defined by

$$Su^\varepsilon = \frac{1}{2} ((\nabla u^\varepsilon) + (\nabla u^\varepsilon)^t).$$

For smooth solutions, it is noticed that

$$(2S(v)n - (\nabla \times v) \times n)_\tau = -(2S(n)v)_\tau,$$

see [6] for detail. Hence, the boundary condition (1.2) can be written in the form of the vorticity as

$$u^\varepsilon \cdot n = 0, \quad n \times (\nabla \times u^\varepsilon) = [Bu^\varepsilon]_\tau, \quad \text{and} \quad \frac{\partial d^\varepsilon}{\partial n} = 0, \quad \text{on } \partial\Omega, \quad (1.3)$$

where  $B = 2(A - S(n))$  is symmetric matrix. Actually, it turns out that the form (1.3) will be more convenient than (1.2) in the energy estimates.

In this paper, we are interested in the existence of strong solution of (1.1) with uniform bounds on an interval of time independent of viscosity  $\varepsilon \in (0, 1]$  and the vanishing viscosity limit to the corresponding inviscid nematic liquid crystal flows as  $\varepsilon$  vanishes, i.e.

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = -\nabla d \cdot \Delta d, & (x, t) \in \Omega \times (0, T), \\ d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, & (x, t) \in \Omega \times (0, T), \\ \operatorname{div} u = 0, & (x, t) \in \Omega \times (0, T), \end{cases} \quad (1.4)$$

with the boundary condition

$$u \cdot n = 0, \quad \text{and} \quad \frac{\partial d}{\partial n} = 0, \quad \text{on } \partial\Omega. \quad (1.5)$$

When  $d$  is a constant vector field, the systems (1.1) and (1.4) are the well-known Navier-Stokes equations and Euler equations respectively. There is lots of literature on the uniform bounds and the vanishing viscosity limit for the Navier-Stokes equations when the domain has no boundaries, for instance, [7–10]. The problem is that due to the presence of a boundary the time of existence  $T^\varepsilon$  depends on the viscosity, and one often cannot prove that it stays bounded away from zero. Nevertheless, in a domain with boundaries, for some special types of Navier-slip boundary conditions or boundaries, some uniform  $H^3$  (or  $W^{2,p}$ , with  $p$  large enough) estimates and a uniform time of existence for Navier-Stokes when the viscosity goes to zero have recently been obtained (see [11–13]). It is easy to see that, for these special boundary conditions, the main part of the boundary layer vanishes, which allows this uniform control in some limited regularity Sobolev space. Recently, Masmoudi and Rousset [14] established conormal uniform estimates for three-dimensional general

smooth domains with the Navier-slip boundary condition, which, in particular, implies the uniform boundedness of the normal first order derivatives of the velocity field. This allows the authors [14] to obtain the convergence of the viscous solutions to the inviscid ones by a compact argument. Based on the uniform estimates in [14], better convergence with rates have been studied in [5] and [15]. In particular, Xiao and Xin [15] have proved the convergence in  $L^\infty(0, T; H^1)$  with a rate of convergence. Recently, Wang et al. [16] investigated the inviscid limit for isentropic compressible Navier-Stokes equation with general Navier-slip boundary conditions. The authors in [16] not only proved that solution is uniformly bounded in a conormal Sobolev space and is uniformly bounded in  $W^{1,\infty}$ , but also obtained the convergence rate of viscous solutions to the inviscid ones. As the vanishing viscosity limit for the full compressible Navier-Stokes equations with general Navier-slip and Neumann boundary conditions, the readers can refer to [17].

Before stating our main results, we first explain the notations and conventions used throughout this paper. Similar to [14, 16], one assumes that the bounded domain  $\Omega \subset \mathbb{R}^3$  has a covering that

$$\Omega \subset \Omega_0 \cup_{k=1}^n \Omega_k, \quad (1.6)$$

where  $\overline{\Omega}_0$ , and in each  $\Omega_k$  there exists a function  $\psi_k$  such that

$$\begin{aligned} \Omega \cap \Omega_k &= \{x = (x_1, x_2, x_3) | x_3 > \psi_k(x_1, x_2)\} \cap \Omega_k, \\ \partial\Omega \cap \Omega_k &= \{x_3 = \psi_k(x_1, x_2)\} \cap \Omega_k. \end{aligned}$$

Here,  $\Omega$  is said to be  $\mathcal{C}^m$  if the functions  $\psi_k$  are a  $\mathcal{C}^m$ -function. To define the conormal Sobolev spaces, one considers  $(Z_k)_{1 \leq k \leq N}$  to be a finite set of generators of vector fields that are tangential to  $\partial\Omega$ , and sets

$$H_{co}^m = \{f \in L^2(\Omega) | Z^I f \in L^2(\Omega), \text{ for } |I| \leq m\},$$

where  $I = (k_1, \dots, k_m)$ . The following notations will be used

$$\begin{aligned} \|u\|_m^2 &= \|u\|_{H_{co}^m}^2 = \sum_{j=1}^3 \sum_{|I| \leq m} \|Z^I u_j\|_{L^2}^2, \\ \|u\|_{m,\infty}^2 &= \sum_{|I| \leq m} \|Z^I u\|_{L^\infty}^2, \end{aligned}$$

and

$$\|\nabla Z^m u\|^2 = \sum_{|I|=m} \|\nabla Z^I u\|_{L^2}^2.$$

Noting that by using the covering of  $\Omega$ , one can always assume that each vector field  $(u, d)$  is supported in one of the  $\Omega_i$ , and moreover, in  $\Omega_0$  the norm  $\|\cdot\|_m$  yields a control of the standard  $H^m$  norm, whereas if  $\Omega_i \cap \partial\Omega \neq \emptyset$ , there is no control of the normal derivatives.

Since  $\partial\Omega$  is given locally by  $x_3 = \psi(x_1, x_2)$  (we omit the subscript  $j$  of notational convenience), it is convenient to use the coordinates

$$\Psi : (y, z) \mapsto (y, \psi(y) + z) = x.$$

A basis is thus given by the vector fields  $(e_{y^1}, e_{y^2}, e_z)$ , where  $e_{y^1} = (1, 0, \partial_1 \psi)^t$ ,  $e_{y^2} = (0, 1, \partial_2 \psi)^t$ , and  $e_z = (0, 0, -1)^t$ . On the boundary,  $e_{y^1}$  and  $e_{y^2}$  are tangent to  $\partial\Omega$ , and in general,  $e_z$  is not a

normal vector field. By using this parametrization, one can take as suitable vector fields compactly supported in  $\Omega_j$  in the definition of the  $\|\cdot\|_m$  norms

$$Z_i = \partial_{y^i} = \partial_i + \partial_i \psi \partial_z, \quad i = 1, 2, \quad Z_3 = \varphi(z) \partial_z,$$

where  $\varphi(z) = \frac{z}{1+z}$  is smooth, supported in  $\mathbb{R}_+$  with the property  $\varphi(0) = 0, \varphi'(0) > 0, \varphi(z) > 0$  for  $z > 0$ . It is easy to check that

$$Z_k Z_j = Z_j Z_k, \quad j, k = 1, 2, 3,$$

and

$$\partial_z Z_i = Z_i \partial_z, \quad i = 1, 2; \quad \partial_z Z_3 \neq Z_3 \partial_z.$$

We shall still denote by  $\partial_j$ ,  $j = 1, 2, 3$ , or  $\nabla$  the derivatives in the physical space. The coordinates of a vector field  $u$  in the basis  $(e_{y^1}, e_{y^2}, e_z)$  will be denoted by  $u^i$ , and thus

$$u = u^1 e_{y^1} + u^2 e_{y^2} + u^3 e_z.$$

We shall denote by  $u_j$  the coordinates in the standard basis of  $\mathbb{R}^3$ , i.e.  $u = u_1 e_1 + u_2 e_2 + u_3 e_3$ . Denote by  $n$  the unit outward normal in the physical space which is given locally by

$$n(x) \equiv n(\Psi(y, z)) = \frac{1}{\sqrt{1 + |\nabla \psi(y)|^2}} \begin{pmatrix} \partial_1 \psi(y) \\ \partial_2 \psi(y) \\ -1 \end{pmatrix} \triangleq \frac{-N(y)}{\sqrt{1 + |\nabla \psi(y)|^2}}, \quad (1.7)$$

and by  $\Pi$  the orthogonal projection

$$\Pi u \equiv \Pi(\Psi(y, z))u = u - [u \cdot n(\Psi(y, z))]n(\Psi(y, z)), \quad (1.8)$$

which gives the orthogonal projection on to the tangent space of the boundary. Note that  $n$  and  $\Pi$  are defined in the whole  $\Omega_k$  and do not depend on  $z$ . For later use and notational convenience, set

$$\mathcal{Z}^\alpha = \partial_t^{\alpha_0} Z^{\alpha_1} = \partial_t^{\alpha_0} Z_1^{\alpha_{11}} Z_2^{\alpha_{12}} Z_3^{\alpha_{13}},$$

where  $\alpha, \alpha_0$  and  $\alpha_1$  are the differential multi-indices with  $\alpha = (\alpha_0, \alpha_1), \alpha_1 = (\alpha_{11}, \alpha_{12}, \alpha_{13})$  and we also use the notation

$$\|f(t)\|_{\mathcal{H}^m}^2 = \sum_{|\alpha| \leq m} \|\mathcal{Z}^\alpha f(t)\|_{L_x^2}^2, \quad (1.9)$$

and

$$\|f(t)\|_{\mathcal{H}^{k,\infty}} = \sum_{|\alpha| \leq k} \|\mathcal{Z}^\alpha f(t)\|_{L_x^\infty} \quad (1.10)$$

for smooth space-time function  $f(x, t)$ . Throughout this paper, the positive generic constants that are independent of  $\varepsilon$  are denoted by  $c, C$ . Denote by  $C_k$  a positive constant independent of  $\varepsilon \in (0, 1]$  which depends only on the  $\mathcal{C}^k$ -norm of the functions  $\psi_j$ ,  $j = 1, \dots, n$ .  $\|\cdot\|$  denotes the standard  $L^2(\Omega; dx)$  norm, and  $\|\cdot\|_{H^m}$  ( $m = 1, 2, 3, \dots$ ) denotes the Sobolev  $H^m(\Omega, dx)$  norm. The notation  $|\cdot|_{H^m}$  will be used for the standard Sobolev norm of functions defined on  $\partial\Omega$ . Note that this norm involves only tangential derivatives.  $P(\cdot)$  denotes a polynomial function.

Since the boundary layers may appear in the presence of physical boundaries, in order to obtain the uniform estimation for solutions to the nematic liquid crystal flows with Navier-slip and Neumann boundary conditions, we need to find a suitable functional space. Indeed, it is impossible

to get a uniform bound for  $(u, d)$  in a standard Sobolev spaces due to possible boundary layers. In order to overcome such a difficulty, Masmoudi and Rousset [14] used conormal Sobolev space to investigate the uniform regularity of the solution for the incompressible Navier-Stokes equations. More precisely, they defined the functional space

$$\sup_{0 \leq t \leq T} \{ \|u(t)\|_{H_{co}^m}^2 + \|\nabla u(t)\|_{H_{co}^{m-1}}^2 + \|\nabla u\|_{1,\infty}^2 \},$$

which implies that the solutions are uniform bounded in  $W^{1,\infty}$ . Note that the nematic liquid crystal flows (1.1) is a coupling between the incompressible Navier-Stokes equations and a transported heat flow of harmonic maps into  $S^2$ . Then, in the spirit of Masmoudi and Rousset [14], we also investigate the solutions of the nematic liquid crystal flows in Conormal Sobolev space. Hence, the functional space should include some information for the direction field  $d$ . On the other hand, due to the nonlinear higher order derivatives term  $\nabla d \cdot \Delta d$ , one should control this term by using the dissipative term  $\Delta d$  on the right hand side of the equation (1.1)<sub>2</sub> which involving the time derivatives term  $d_t$ . Hence, we also include some information involving the time derivatives in the functional space. Therefore, we define the functional space  $X_m(T)$  for a pair of function  $(u, d)(x, t)$  as follows

$$X_m(T) = \{(u, d) \in L^\infty([0, T], L^2); \operatorname{esssup}_{0 \leq t \leq T} \|(u, d)(t)\|_{X_m} < +\infty\}, \quad (1.11)$$

where the norm  $\|(\cdot, \cdot)\|_{X_m}$  is given by

$$\|(u, d)(t)\|_{X_m} = \|u\|_{\mathcal{H}^m}^2 + \|d\|_{L^2}^2 + \|\nabla d\|_{\mathcal{H}^m}^2 + \|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|\Delta d\|_{\mathcal{H}^{m-1}}^2 + \|\nabla u\|_{\mathcal{H}^{1,\infty}}^2. \quad (1.12)$$

In the present paper, we supplement the nematic liquid crystal flows system (1.1) with initial data

$$(u^\varepsilon, d^\varepsilon)(x, 0) = (u_0^\varepsilon, d_0^\varepsilon)(x), \quad (1.13)$$

and

$$\begin{aligned} \sup_{0 < \varepsilon \leq 1} \|(u_0^\varepsilon, d_0^\varepsilon)\|_{X_m} &= \sup_{0 < \varepsilon \leq 1} \{ \|u_0^\varepsilon\|_{\mathcal{H}^m}^2 + \|d_0^\varepsilon\|_{L^2}^2 + \|\nabla d_0^\varepsilon\|_{\mathcal{H}^m}^2 + \|\nabla u_0^\varepsilon\|_{\mathcal{H}^{m-1}}^2 \\ &\quad + \|\Delta d_0^\varepsilon\|_{\mathcal{H}^{m-1}}^2 + \|\nabla u_0^\varepsilon\|_{\mathcal{H}^{1,\infty}}^2 \} \leq \tilde{C}_0, \end{aligned} \quad (1.14)$$

where  $\tilde{C}_0$  is a positive constant independent of  $\varepsilon \in (0, 1]$ , and the time derivatives of initial data are defined through the equation (1.1). Thus, the initial data  $(u_0^\varepsilon, d_0^\varepsilon)$  is assumed to have a higher space regularity and compatibilities. Notice that the a priori estimates in Theorem 3.1 below are obtained in the case that the approximate solution is sufficiently smooth up to the boundary, and therefore, in order to obtain a selfconstained result, one needs to assume the approximated initial data satisfies the boundary compatibilities condition (1.3). For the initial data  $(u_0^\varepsilon, d_0^\varepsilon)$  satisfying (1.13), it is not clear if there exists an approximate sequences  $(u_0^{\varepsilon,\delta}, d_0^{\varepsilon,\delta})$  ( $\delta$  being a regularization parameter) which satisfy the boundary compatibilities and  $\|(u_0^{\varepsilon,\delta} - u_0^\varepsilon, d_0^{\varepsilon,\delta} - d_0^\varepsilon)\|_{X_m} \rightarrow 0$  as  $\delta \rightarrow 0$ . Therefore, we set

$$\begin{aligned} X_{n,m}^{ap} &= \left\{ (u, d) \in H^{4m}(\Omega) \times H^{4(m+1)}(\Omega) \mid \partial_t^k u, \partial_t^k d, \partial_t^k \nabla d, k = 1, \dots, m \text{ are defined through} \right. \\ &\quad \left. \text{equations (1.1) the and } \partial_t^k u, \partial_t^k \nabla d, k = 0, \dots, m-1 \right. \\ &\quad \left. \text{satisfy the boundary compatibility condition} \right\} \end{aligned} \quad (1.15)$$

and

$$X_{n,m} = \text{the closure of } X_{n,m}^{ap} \text{ in the norm } \|(\cdot, \cdot)\|_{X_m}. \quad (1.16)$$

Now, we state the first results concerning the uniform regularity for the nematic liquid crystal flows (1.1), (1.3) and (1.13) as follows.

**Theorem 1.1 (Uniform Regularity).** *Let  $m$  be an integer satisfying  $m \geq 6$ ,  $\Omega$  be a  $C^{m+2}$  domain, and  $A \in C^{m+1}(\partial\Omega)$ . Consider the initial data  $(u_0^\varepsilon, d_0^\varepsilon) \in X_{n,m}$  satisfy (1.14) and  $\operatorname{div} u_0^\varepsilon = 0$ ,  $|d_0^\varepsilon| = 1$  in  $\overline{\Omega}$ . Then, there exists a time  $T_0 > 0$  and  $\tilde{C}_1 > 0$  independent of  $\varepsilon \in (0, 1]$ , such that there exists a unique solution of (1.1), (1.3) and (1.13) which is defined on  $[0, T_0]$  and satisfies the estimate*

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} (\|d^\varepsilon\|_{L^2}^2 + \|(u^\varepsilon, \nabla d^\varepsilon)\|_{\mathcal{H}^m}^2 + \|(\nabla u^\varepsilon, \Delta d^\varepsilon)\|_{\mathcal{H}^{m-1}}^2 + \|\nabla u^\varepsilon\|_{\mathcal{H}^{1,\infty}}^2) \\ & + \varepsilon \int_0^t (\|\nabla u^\varepsilon\|_{\mathcal{H}^m}^2 + \|\nabla^2 u^\varepsilon\|_{\mathcal{H}^{m-1}}^2) d\tau + \int_0^t (\|\Delta d^\varepsilon\|_{\mathcal{H}^m}^2 + \|\nabla \Delta d^\varepsilon\|_{\mathcal{H}^{m-1}}^2) d\tau \leq \tilde{C}_1, \end{aligned} \quad (1.17)$$

where  $\tilde{C}_1$  depends only on  $\tilde{C}_0$  and  $C_{m+2}$ .

**Remark 1.1.** For  $(u_0^\varepsilon, d_0^\varepsilon) \in X_{n,m}$ , it must hold that  $u_0^\varepsilon \cdot n|_{\partial\Omega} = 0$ ,  $((Su_0^\varepsilon)n)_\tau|_{\partial\Omega} = -(Au_0^\varepsilon)_\tau|_{\partial\Omega}$ , and  $n \cdot \nabla d_0^\varepsilon|_{\partial\Omega} = 0$  in the trace sense for every fixed  $\varepsilon \in (0, 1]$ .

**Remark 1.2.** Indeed, the nematic liquid crystal flows (1.1) has the following form

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = \varepsilon \Delta u - \lambda \nabla d \cdot \Delta d, & (x, t) \in \Omega \times (0, T), \\ d_t + u \cdot \nabla d = \theta(\Delta d + |\nabla d|^2 d), & (x, t) \in \Omega \times (0, T), \\ \operatorname{div} u = 0, & (x, t) \in \Omega \times (0, T). \end{cases} \quad (1.18)$$

Here the parameters  $\varepsilon$ ,  $\lambda$  and  $\theta$  are positive constants representing the fluid viscosity, the competition between kinetic energy and potential energy, and the macroscopic elastic relaxation time for the molecular orientation field respectively. If  $\theta = 0$ , system (1.18) for  $(u, d)$  is closely related to the MHD system for  $(u, \psi)$  provided we identify  $\psi = \nabla \times d$ . There have been many interesting works on global small solutions to the MHD system recently, refer to [18–20]. Due to the nonlinear term  $\lambda \nabla d \cdot \Delta d$  on the right hand side of (1.18)<sub>1</sub>, we can not obtain the uniform regularity estimates for the case that both  $\varepsilon$  and  $\theta$  go to zero in this paper.

The main steps of the proof of Theorem 1.1 are the following. First, we obtained a conormal energy estimates for  $(u^\varepsilon, \nabla d^\varepsilon)$  in  $\mathcal{H}^m$ -norm (see (1.9) above for the definition of  $\mathcal{H}^m$ ). The second step is to give the estimates for  $\|\Delta d^\varepsilon\|_{\mathcal{H}^{m-1}}$ , which is easy to obtain since there exists a dissipative term  $\Delta d^\varepsilon$  on the right-hand side of (1.1)<sub>2</sub>. The third step is to give the estimate for  $\|\partial_n u^\varepsilon\|_{\mathcal{H}^{m-1}}$  and  $\|\Delta d^\varepsilon\|_{\mathcal{H}^{m-1}}$ . In order to obtain this estimate by an energy method,  $\partial_n u^\varepsilon$  is not a convenient quantity because it does not vanish on the boundary. Similar to [14], the quantity  $\partial_n u^\varepsilon \cdot n$  can be controlled thanks to the divergence free condition of  $u^\varepsilon$ . In order to give the estimate for  $(\partial_n u^\varepsilon)_\tau$ , one choose the convenient quantity  $\eta = w^\varepsilon \times n + (Bu^\varepsilon)_\tau$  with a homogeneous Dirichlet boundary conditions. The fourth step is to give the estimate for the pressure, which is not transparent in the estimate since the conormal fields  $Z_i$  do not communicate with the gradient operator. In the spirit of [14], we split the pressure into two parts, the first part have the same regularity as in the Euler equation and the second part is linked to the Navier condition. The fifth step is to estimate

$\|\Delta d^\varepsilon\|_{W^{1,\infty}}$ . Indeed, this estimate is easy to obtain since there exists a dissipation term  $\Delta d^\varepsilon$  on the right-hand side of (1.1)<sub>2</sub>. The last step is to estimate  $\|\nabla u^\varepsilon\|_{\mathcal{H}^{1,\infty}}$ . In fact, it suffices to estimate  $\|(\partial_n u^\varepsilon)_\tau\|_{\mathcal{H}^{1,\infty}}$  since the other terms can be estimated by the Sobolev embedding. We choose an equivalent quantity such that it satisfies a homogeneous Dirichlet condition and solves a convection-diffusion equation at the leading order. Then Theorem 1.1 can be proved by these a priori estimates and a classical iteration method.

Based on the uniform estimates in Theorem 1.1, in the spirit of similar arguments [14], we can obtain the vanishing viscosity limit of solutions of (1.1) to the solutions of (1.4) in  $L^\infty$ -norm by the strong compactness argument, without convergence rate. In this paper, we hope to prove the vanishing viscosity limit with rates of convergence, which can be stated as follows.

**Theorem 1.2 (Inviscid Limit).** *Let  $(u, d)(t) \in L^\infty(0, T_1; H^3) \times L^\infty(0, T_1; H^4)$  be the smooth solution to the equation (1.4) and (1.3) with initial data  $(u_0, d_0)$  satisfying*

$$(u_0, d_0) \in (H^3 \times H^4) \cap X_{n,m} \text{ with } m \geq 6. \quad (1.19)$$

*Let  $(u^\varepsilon, d^\varepsilon)(t)$  be the solution to the initial boundary value problem of the nematic liquid crystal flows (1.1), (1.2) with initial data  $(u_0, d_0)$  satisfying (1.19). Then, there exists  $T_2 = \min\{T_0, T_1\} > 0$ , which is independent of  $\varepsilon > 0$ , such that*

$$\|(u^\varepsilon - u)(t)\|_{L^2}^2 + \|(d^\varepsilon - d)(t)\|_{H^1}^2 \leq C\varepsilon^{\frac{3}{2}}, \quad t \in [0, T_2] \quad (1.20)$$

and

$$\|(u^\varepsilon - u)\|_{L^\infty(0, T_2; L^\infty(\Omega))} + \|(d^\varepsilon - d)\|_{L^\infty(0, T_2; W^{1,\infty}(\Omega))} \leq C\varepsilon^{\frac{3}{10}}, \quad (1.21)$$

which  $C$  depends only on the norm  $\|u_0\|_{H^3}, \|d_0\|_{H^4}$  and  $\|(u_0, d_0)\|_{X_m}$ .

The rest of the paper is organized as follows: In section 2, we collect some inequalities that will be used later. In section 3, the a priori estimates in Theorem 3.1 are proved. By using these a priori estimates, one give the proof for the Theorem 1.1 in section 4. Based on the uniform estimates obtained in Theorem 1.1, we establish the convergence rate for the solutions from (1.1) to (1.4) and complete the proof for the Theorem 1.2.

## 2 Preliminaries

The following lemma [13, 21] allows us to control the  $H^m(\Omega)$ -norm of a vector valued function  $u$  by its  $H^{m-1}$ -norm of  $\nabla \times u$  and  $\operatorname{div} u$ , together with the  $H^{m-\frac{1}{2}}(\partial\Omega)$  of  $u \cdot n$ .

**Proposition 2.1.** *Let  $m \in \mathbb{N}_+$  be an integer. Let  $u \in H^m$  be a vector-valued function. Then, there exists a constant  $C > 0$  independent of  $u$ , such that*

$$\|u\|_{H^m} \leq C(\|\nabla \times u\|_{H^{m-1}} + \|\operatorname{div} u\|_{H^{m-1}} + \|u\|_{H^{m-1}} + |u \cdot n|_{H^{m-\frac{1}{2}}(\partial\Omega)}), \quad (2.1)$$

and

$$\|u\|_{H^m} \leq C(\|\nabla \times u\|_{H^{m-1}} + \|\operatorname{div} u\|_{H^{m-1}} + \|u\|_{H^{m-1}} + |n \times u|_{H^{m-\frac{1}{2}}(\partial\Omega)}). \quad (2.2)$$

In this paper, one repeatedly use the Gagliardo-Nirenberg-Morser type inequality, whose proof can be find in [22]. First, define the space

$$W^m(\Omega \times [0, T]) = \{f(x, t) \in L^2(\Omega \times [0, T]) | \mathcal{Z}^\alpha f \in L^2(\Omega \times [0, T]), |\alpha| \leq m\}. \quad (2.3)$$

Then, the Gagliardo-Nirenberg-Morser type inequality can be stated as follows:

**Proposition 2.2.** *For  $u, v \in L^\infty(\Omega \times [0, T]) \cap \mathcal{W}^m(\Omega \times [0, T])$  with  $m \in \mathbb{N}_+$  an integer, it holds that*

$$\int_0^t \|(\mathcal{Z}^\beta u \mathcal{Z}^\gamma v)(\tau)\|^2 d\tau \lesssim \|u\|_{L_{t,x}^\infty}^2 \int_0^t \|v(\tau)\|_{\mathcal{H}^m}^2 d\tau + \|v\|_{L_{t,x}^\infty}^2 \int_0^t \|u(\tau)\|_{\mathcal{H}^m}^2 d\tau, \quad |\beta| + |\gamma| = m. \quad (2.4)$$

We also need the following anisotropic Sobolev embedding and trace theorems, refer to [16, 17].

**Proposition 2.3.** *Let  $m_1 \geq 0, m_2 \geq 0$  be integers and  $f \in H_{co}^{m_1}(\Omega) \cap H_{co}^{m_2}(\Omega)$  and  $\nabla f \in H_{co}^{m_2}(\Omega)$ .*

(1) *The following anisotropic Sobolev embedding holds:*

$$\|f\|_{L^\infty}^2 \leq C(\|\nabla f\|_{H_{co}^{m_2}} + \|f\|_{H_{co}^{m_2}}) \cdot \|f\|_{H_{co}^{m_1}}, \quad (2.5)$$

*provided  $m_1 + m_2 \geq 3$ .*

(2) *The following trace estimate holds:*

$$\|f\|_{H^s(\partial\Omega)}^2 \leq C(\|\nabla f\|_{H_{co}^{m_2}} + \|f\|_{H_{co}^{m_2}}) \cdot \|f\|_{H_{co}^{m_1}}, \quad (2.6)$$

*provided  $m_1 + m_2 \geq 2s \geq 0$ .*

### 3 A priori estimates

The aim of this section is to prove the following a priori estimates, which are crucial to prove Theorem 1.1. For notational convenience, we drop the superscript  $\varepsilon$  throughout this section.

**Theorem 3.1** (a priori estimates). *Let  $m$  be an integer satisfying  $m \geq 6$ ,  $\Omega$  be a  $C^{m+2}$  domain, and  $A \in \mathcal{C}^{m+1}(\partial\Omega)$ . For sufficiently smooth solutions defined on  $[0, T]$  of (1.1) and (1.2), then the following a priori estimate holds*

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} N_m(\tau) + \varepsilon \int_0^t (\|\nabla u\|_{\mathcal{H}^m}^2 + \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2) d\tau + \int_0^t (\|\Delta d\|_{\mathcal{H}^m}^2 + \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2) d\tau \\ & \leq \tilde{C}_2 C_{m+2} \left\{ N_m(0) + P(N_m(t)) \int_0^t P(N_m(\tau)) d\tau \right\}, \quad \forall t \in [0, T], \end{aligned} \quad (3.1)$$

where  $\tilde{C}_2$  depends only on  $\frac{1}{c_0}$ ,  $P(\cdot)$  is a polynomial, and

$$N_m(t) \triangleq \|u\|_{\mathcal{H}^m}^2 + \|d\|_{L^2}^2 + \|\nabla d\|_{\mathcal{H}^m}^2 + \|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|\Delta d\|_{\mathcal{H}^{m-1}}^2 + \|\nabla u\|_{\mathcal{H}^{1,\infty}}^2. \quad (3.2)$$

Since the proof of Theorem 3.1 is quite lengthy and involved, we divide the proof into the following several subsections.



### 3.1. Conormal Estimates for $u$ and $\nabla d$

For any smooth function  $f$ , notice that

$$\Delta f = \nabla \operatorname{div} f - \nabla \times (\nabla \times f),$$

and then (1.1)<sub>1</sub> can be written as

$$u_t + u \cdot \nabla u + \nabla p = -\varepsilon \nabla \times (\nabla \times u) - \nabla d \cdot \Delta d. \quad (3.3)$$

In this subsection, we first give the basic a priori  $L^2$  energy estimate which holds for (1.1) and (1.3). Let

$$Q(t) \triangleq \sup_{0 \leq \tau \leq t} \{ \| (u, u_\tau)(\tau) \|_{L^\infty}^2 + \| d_\tau(\tau) \|_{W^{1,\infty}}^2 + \| \nabla d(\tau) \|_{W^{1,\infty}}^2 + \| \nabla \Delta d(\tau) \|_{L^\infty}^2 + \| \nabla u(\tau) \|_{\mathcal{H}^{1,\infty}}^2 \}. \quad (3.4)$$

**Lemma 3.2.** *For a smooth solution to (1.1) and (1.3), it holds that for  $\varepsilon \in (0, 1]$*

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} \int (|u|^2 + |d|^2 + |\nabla d|^2)(\tau) dx + \varepsilon \int_0^t \int |\nabla u|^2 dx d\tau + \int_0^t \int |\Delta d|^2 dx d\tau \\ & \leq \int (|u_0|^2 + |\nabla d_0|^2) dx + C_2[1 + Q(t)] \int_0^t N_0(\tau) d\tau. \end{aligned} \quad (3.5)$$

*Proof.* Multiplying (1.1)<sub>2</sub> by  $d$  and integrating over  $\Omega$ , one arrives at

$$\frac{d}{dt} \frac{1}{2} \int (|d|^2 - 1) dx + \int u \cdot \nabla (|d|^2 - 1) dx = \int \Delta d \cdot d dx + \int |\nabla d|^2 |d|^2 dx,$$

which, integrating by part and applying the boundary condition (1.3), yields that

$$\frac{d}{dt} \int (|d|^2 - 1) dx + 2 \int (|d|^2 - 1) |\nabla d|^2 dx = 0. \quad (3.6)$$

In view of the Grönwall inequality, one deduce from the identity (3.6) that

$$|d| = 1 \quad \text{in } \overline{\Omega}. \quad (3.7)$$

Multiplying (3.3) by  $u$ , integrating by parts and applying the boundary condition (1.3), we find

$$\frac{1}{2} \frac{d}{dt} \int |u|^2 dx + \varepsilon \int \nabla \times (\nabla \times u) \cdot u dx = - \int (u \cdot \nabla) d \cdot \Delta d dx. \quad (3.8)$$

Integrating by part and applying the boundary condition (1.3), we get

$$\begin{aligned} \int \nabla \times (\nabla \times u) u dx &= \int_{\partial\Omega} n \times (\nabla \times u) u d\sigma + \int |\nabla \times u|^2 dx \\ &= \int_{\partial\Omega} [Bu]_\tau \cdot u_\tau d\sigma + \int |\nabla \times u|^2 dx, \end{aligned}$$

which, together with (3.8), gives directly

$$\frac{1}{2} \frac{d}{dt} \int |u|^2 dx + \varepsilon \int |\nabla \times u|^2 dx = -\varepsilon \int_{\partial\Omega} [Bu]_\tau \cdot u_\tau d\sigma - \int (u \cdot \nabla) d \cdot \Delta d dx. \quad (3.9)$$

Multiplying (1.1)<sub>3</sub> by  $\Delta d$  and integrating over  $\Omega$ , one arrives at

$$\int (d_t + u \cdot \nabla d) \cdot \Delta d \, dx = \int |\Delta d|^2 dx + \int |\nabla d|^2 d \cdot \Delta d \, dx. \quad (3.10)$$

Integration by part and application of boundary condition (1.3) yield directly

$$\int d_t \cdot \Delta d \, dx = \int_{\partial\Omega} d_t \cdot \frac{\partial d}{\partial n} d\sigma - \frac{1}{2} \frac{d}{dt} \int |\nabla d|^2 dx = -\frac{1}{2} \frac{d}{dt} \int |\nabla d|^2 dx. \quad (3.11)$$

By virtue of the basic fact  $|d| = 1$ , we find  $\Delta d \cdot d = -|\nabla d|^2$ . Then, the combination of (3.10) and (3.11) gives

$$\frac{1}{2} \frac{d}{dt} \int |\nabla d|^2 dx + \int |\Delta d|^2 dx = \int (u \cdot \nabla) d \cdot \Delta d \, dx + \int |\nabla d|^4 dx,$$

which, together with (3.9), yields directly

$$\frac{1}{2} \frac{d}{dt} \int (|u|^2 + |\nabla d|^2) dx + \varepsilon \int |\nabla \times u|^2 dx + \int |\Delta d|^2 dx \leq \int |\nabla d|^4 dx + \varepsilon C_2 \int_{\partial\Omega} |u|^2 d\sigma. \quad (3.12)$$

The trace theorem in Proposition 2.3 implies

$$|u|_{L^2(\partial\Omega)}^2 \leq \delta \|\nabla u\|^2 + C_\delta \|u\|^2. \quad (3.13)$$

The application of Proposition 2.1 gives immediately

$$\|\nabla \times u\|_{L^2}^2 \geq c_1 \|\nabla u\|_{L^2}^2 - c_2 \|u\|_{L^2}^2. \quad (3.14)$$

Substituting (3.13) and (3.14) into (3.12) and choosing  $\delta$  small enough, one arrives at

$$\frac{d}{dt} \int (|u|^2 + |\nabla d|^2) dx + \varepsilon \int |\nabla u|^2 dx + \int |\Delta d|^2 dx \leq \int |\nabla d|^4 dx + C_2 \int |u|^2 dx,$$

which, integrating over  $[0, t]$ , yields

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} \int (|u|^2 + |\nabla d|^2)(\tau) dx + \varepsilon \int_0^t \int |\nabla u|^2 dx d\tau + \int_0^t \int |\Delta d|^2 dx d\tau \\ & \leq \int (|u_0|^2 + |\nabla d_0|^2) dx + \|\nabla d\|_{L_{x,t}^\infty}^2 \int_0^t \int |\nabla d|^2 dx d\tau + C_2 \int_0^t \int |u|^2 dx d\tau. \end{aligned}$$

Therefore, we complete the proof of the lemma 3.2.  $\square$

However, the above basic energy estimation is insufficient to get the vanishing viscosity limit. Some conormal derivative estimates are needed.

**Lemma 3.3.** *For a smooth solution to (1.1) and (1.3), it holds that for every  $m \in \mathbb{N}_+$  and  $\varepsilon \in (0, 1]$*

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} \|(u, \nabla d)\|_{\mathcal{H}^m}^2 + \varepsilon \int_0^t \|\nabla u\|_{\mathcal{H}^m}^2 d\tau + \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau \\ & \leq C_{m+2} \left\{ \|(u_0, \nabla d_0)\|_{\mathcal{H}^m}^2 + \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + \delta \varepsilon^2 \int \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau \right. \\ & \quad \left. + \int_0^t (\|\nabla^2 p_1\|_{\mathcal{H}^{m-1}} \|u\|_{\mathcal{H}^m} + \varepsilon^{-1} \|\nabla p_2\|_{\mathcal{H}^{m-1}}^2) d\tau + C_\delta [1 + P(Q(t))] \int_0^t N_m(\tau) d\tau \right\}, \end{aligned} \quad (3.15)$$

where the pressure  $p$  is split as  $p = p_1 + p_2$ , where  $p_1$  and  $p_2$  are defined by (3.29) and (3.30) respectively.

*Proof.* The case for  $m = 0$  is already proved in Lemma 3.2. Assume that (3.15) is proved for  $k = m - 1$ . We shall prove that holds for  $k = m \geq 1$ . Applying the operator  $\mathcal{Z}^\alpha (|\alpha_0| + |\alpha_1| = m)$  to the equation (3.3), we find

$$\mathcal{Z}^\alpha u_t + u \cdot \nabla \mathcal{Z}^\alpha u + \mathcal{Z}^\alpha \nabla p = -\varepsilon \mathcal{Z}^\alpha \nabla \times (\nabla \times u) - \mathcal{Z}^\alpha (\nabla d \cdot \Delta d) + \mathcal{C}_1^\alpha, \quad (3.16)$$

where

$$\mathcal{C}_1^\alpha = -[\mathcal{Z}^\alpha, u \cdot \nabla]u.$$

Multiplying (3.16) by  $\mathcal{Z}^\alpha u$ , it is easy to deduce that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |\mathcal{Z}^\alpha u|^2 dx + \int \mathcal{Z}^\alpha \nabla p \cdot \mathcal{Z}^\alpha u dx \\ &= -\varepsilon \int \mathcal{Z}^\alpha \nabla \times (\nabla \times u) \cdot \mathcal{Z}^\alpha u dx - \int \mathcal{Z}^\alpha (\nabla d \cdot \Delta d) \cdot \mathcal{Z}^\alpha u dx + \int \mathcal{C}_1^\alpha \cdot \mathcal{Z}^\alpha u dx. \end{aligned} \quad (3.17)$$

Integrating by part and applying the boundary condition (1.3), one arrives at

$$\begin{aligned} & -\varepsilon \int \mathcal{Z}^\alpha \nabla \times w \cdot \mathcal{Z}^\alpha u dx \\ &= -\varepsilon \int (\nabla \times (\mathcal{Z}^\alpha w) + [\mathcal{Z}^\alpha, \nabla \times]w) \cdot \mathcal{Z}^\alpha u dx \\ &= -\varepsilon \int_{\partial\Omega} n \times \mathcal{Z}^\alpha w \cdot \mathcal{Z}^\alpha u d\sigma - \varepsilon \int \mathcal{Z}^\alpha w \cdot \nabla \times \mathcal{Z}^\alpha u dx - \varepsilon \int [\mathcal{Z}^\alpha, \nabla \times]w \cdot \mathcal{Z}^\alpha u dx \\ &= -\varepsilon \int |\nabla \times \mathcal{Z}^\alpha u|^2 dx - \varepsilon \int [\mathcal{Z}^\alpha, \nabla \times]u \cdot \nabla \times \mathcal{Z}^\alpha u dx - \varepsilon \int [\mathcal{Z}^\alpha, \nabla \times]w \cdot \mathcal{Z}^\alpha u dx \\ & \quad - \varepsilon \int_{\partial\Omega} n \times \mathcal{Z}^\alpha w \cdot \mathcal{Z}^\alpha u d\sigma. \end{aligned} \quad (3.18)$$

Applying the Cauchy inequality, it is easy to deduce that

$$\begin{aligned} & -\varepsilon \int [\mathcal{Z}^\alpha, \nabla \times]u \cdot \nabla \times \mathcal{Z}^\alpha u dx \leq \frac{\varepsilon}{4} \|\nabla \times \mathcal{Z}^\alpha u\|^2 + C \|\nabla u\|_{\mathcal{H}^{m-1}}^2, \\ & -\varepsilon \int [\mathcal{Z}^\alpha, \nabla \times]w \cdot \mathcal{Z}^\alpha u dx \leq \delta \varepsilon^2 \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 + C_\delta \|u\|_{\mathcal{H}^m}^2. \end{aligned} \quad (3.19)$$

To deal with the boundary terms involving  $\mathcal{Z}^\alpha u$  with  $\alpha_{13} = 0$ . (If  $\alpha_{13} \neq 0$ ,  $\mathcal{Z}^\alpha u = 0$  on the boundary by definition). With the help of trace theorem in Proposition 2.3, one has for  $|\alpha_0| + |\alpha_1| = m$

$$\begin{aligned} & |n \times \mathcal{Z}^\alpha w|_{L^2(\partial\Omega)} \\ & \leq C_{m+2} \left( |\partial_t^{\alpha_0} w|_{H^{|\alpha_1|-1}(\partial\Omega)} + |\partial_t^{\alpha_0} u|_{H^{|\alpha_1|}(\partial\Omega)} \right) \\ & \leq C_{m+2} \|\nabla \partial_t^{\alpha_0} w\|_{|\alpha_1|-1}^{\frac{1}{2}} \|\partial_t^{\alpha_0} w\|_{|\alpha_1|-1}^{\frac{1}{2}} + C_{m+2} \|\nabla \partial_t^{\alpha_0} u\|_{|\alpha_1|}^{\frac{1}{2}} \|\partial_t^{\alpha_0} u\|_{|\alpha_1|}^{\frac{1}{2}} \\ & \quad + C_{m+2} (\|\nabla u\|_{\mathcal{H}^{m-1}} + \|u\|_{\mathcal{H}^m}) \\ & \leq C_{m+2} (\|\nabla^2 u\|_{\mathcal{H}^{m-1}}^{\frac{1}{2}} \|\nabla u\|_{\mathcal{H}^{m-1}}^{\frac{1}{2}} + \|\nabla u\|_{\mathcal{H}^m}^{\frac{1}{2}} \|u\|_{\mathcal{H}^m}^{\frac{1}{2}} + \|\nabla u\|_{\mathcal{H}^{m-1}} + \|u\|_{\mathcal{H}^m}) \\ & \leq C_{m+2} (\|\nabla^2 u\|_{\mathcal{H}^{m-1}} + \|\nabla u\|_{\mathcal{H}^{m-1}} + \|\nabla u\|_{\mathcal{H}^m} + \|u\|_{\mathcal{H}^m}). \end{aligned}$$

Note that we use the convention that  $\|\cdot\|_{H^k} = 0$ , for  $k < 0$  here and below. Then the term  $|\partial_t^{\alpha_0} w|_{H^{|\alpha_1|-1}(\partial\Omega)}$  does not show up when  $|\alpha_0| = m$ . Hence, it is easy to deduce that

$$\begin{aligned}
& -\varepsilon \int_{\partial\Omega} n \times \mathcal{Z}^\alpha w \cdot \mathcal{Z}^\alpha u \, d\sigma \\
& \leq \varepsilon |n \times \mathcal{Z}^\alpha w|_{L^2(\partial\Omega)} |\mathcal{Z}^\alpha u|_{L^2(\partial\Omega)} \\
& \leq C_{m+2} \varepsilon (\|\nabla^2 u\|_{\mathcal{H}^{m-1}} + \|\nabla u\|_{\mathcal{H}^{m-1}} + \|\nabla u\|_{\mathcal{H}^m} + \|u\|_{\mathcal{H}^m}) \\
& \quad \times (\|\nabla u\|_{\mathcal{H}^m}^{\frac{1}{2}} + \|u\|_{\mathcal{H}^m}^{\frac{1}{2}}) \|u\|_{\mathcal{H}^m}^{\frac{1}{2}} \\
& \leq \delta \varepsilon \|\nabla u\|_{\mathcal{H}^m}^2 + \delta \varepsilon^2 \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 + C_\delta C_{m+2} (\|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|u\|_{\mathcal{H}^m}^2).
\end{aligned} \tag{3.20}$$

Substituting (3.19) and (3.20) into (3.18), we find

$$\begin{aligned}
& -\varepsilon \int \mathcal{Z}^\alpha \nabla \times (\nabla \times u) \cdot \mathcal{Z}^\alpha u \, dx \\
& \leq -\frac{3\varepsilon}{4} \int |\nabla \times \mathcal{Z}^\alpha u|^2 \, dx + \delta_1 \varepsilon \|\nabla u\|_{\mathcal{H}^m}^2 + \delta_1 \varepsilon^2 \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 \\
& \quad + C_{\delta_1} C_{m+2} (\|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|u\|_{\mathcal{H}^m}^2).
\end{aligned} \tag{3.21}$$

On the other hand, it follows from Proposition 2.1 that

$$\begin{aligned}
\|\nabla \mathcal{Z}^\alpha u\|^2 & \leq C (\|\nabla \times \mathcal{Z}^\alpha u\|^2 + \|\operatorname{div} \mathcal{Z}^\alpha u\|^2 + \|\mathcal{Z}^\alpha u\|^2 + |\mathcal{Z}^\alpha u \cdot n|_{H^{\frac{1}{2}}(\partial\Omega)}) \\
& \leq \|\nabla \times \mathcal{Z}^\alpha u\|^2 + \|\operatorname{div} \mathcal{Z}^\alpha u\|^2 + C_{m+2} (\|u\|_{\mathcal{H}^m}^2 + \|\nabla u\|_{\mathcal{H}^{m-1}}^2),
\end{aligned} \tag{3.22}$$

where we have used the fact that

$$\begin{aligned}
|\mathcal{Z}^\alpha u \cdot n|_{H^{\frac{1}{2}}(\partial\Omega)} & \leq C_{m+2} |\partial_t^{\alpha_0} u|_{H^{m-\alpha_0-\frac{1}{2}}(\partial\Omega)} \\
& \leq C_{m+2} (\|\nabla \partial_t^{\alpha_0} u\|_{m-\alpha_0-1} + \|\partial_t^{\alpha_0} u\|_{m-\alpha_0}) \\
& \leq C_{m+2} (\|\nabla u\|_{\mathcal{H}^{m-1}} + \|u\|_{\mathcal{H}^m}),
\end{aligned}$$

which is a consequence of boundary condition (1.3) and (2.1). The combination of (3.21) and (3.22) gives directly

$$\begin{aligned}
& -\varepsilon \int \mathcal{Z}^\alpha \nabla \times (\nabla \times u) \cdot \mathcal{Z}^\alpha u \, dx \\
& \leq -\frac{3\varepsilon}{4} \int |\nabla \mathcal{Z}^\alpha u|^2 \, dx + \delta_1 \varepsilon \|\nabla u\|_{\mathcal{H}^m}^2 + \delta_1 \varepsilon^2 \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 \\
& \quad + C_{\delta_1} C_{m+2} (\|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|u\|_{\mathcal{H}^m}^2).
\end{aligned} \tag{3.23}$$

Here we have used the divergence free condition of the velocity. Integrating (3.17) over  $[0, t]$  and substituting (3.23) into the resulting identity, one arrives at

$$\begin{aligned}
& \frac{1}{2} \int |\mathcal{Z}^\alpha u|^2 \, dx + \frac{3\varepsilon}{4} \int_0^t \int |\nabla \mathcal{Z}^\alpha u|^2 \, dx \, d\tau \\
& \leq \frac{1}{2} \int |\mathcal{Z}^\alpha u_0|^2 \, dx + \delta_1 \varepsilon \int_0^t \|\nabla u\|_{\mathcal{H}^m}^2 \, d\tau + \delta_1 \varepsilon^2 \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 \, d\tau \\
& \quad + C_{\delta_1} C_{m+2} \int_0^t (\|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|u\|_{\mathcal{H}^m}^2) \, d\tau + \int_0^t \int \mathcal{C}_1^\alpha \cdot \mathcal{Z}^\alpha u \, dx \, d\tau \\
& \quad - \int_0^t \int \mathcal{Z}^\alpha \nabla p \cdot \mathcal{Z}^\alpha u \, dx \, d\tau - \int_0^t \int \mathcal{Z}^\alpha (\nabla d \cdot \Delta d) \cdot \mathcal{Z}^\alpha u \, dx \, d\tau.
\end{aligned} \tag{3.24}$$

To deal with the term  $\int_0^t \int C_1^\alpha \cdot \mathcal{Z}^\alpha u \, dx d\tau$ . Indeed, it is easy to deduce

$$\begin{aligned} \int_0^t \int C_1^\alpha \cdot \mathcal{Z}^\alpha u \, dx d\tau &= - \int_0^t \int ([\mathcal{Z}^\alpha, u] \cdot \nabla u + u \cdot [\mathcal{Z}^\alpha, \nabla] u) \cdot \mathcal{Z}^\alpha u \, dx d\tau \\ &\leq \int_0^t (\|[\mathcal{Z}^\alpha, u] \cdot \nabla u\| + \|u \cdot [\mathcal{Z}^\alpha, \nabla] u\|) \|\mathcal{Z}^\alpha u\| \, d\tau. \end{aligned} \quad (3.25)$$

The help of Proposition 2.2 yields directly

$$\begin{aligned} &\int_0^t \|[\mathcal{Z}^\alpha, u] \cdot \nabla u\|^2 \, d\tau \\ &\leq \sum_{|\beta|+|\gamma|=m, |\beta| \geq 1} C_{\beta, \gamma} \int_0^t \|\mathcal{Z}^\beta u \cdot \mathcal{Z}^\gamma \nabla u\|^2 \, d\tau \\ &\leq \|\mathcal{Z} u\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla u\|_{\mathcal{H}^{m-1}}^2 \, d\tau + C \|\nabla u\|_{L_{x,t}^\infty}^2 \int_0^t \|\mathcal{Z} u\|_{\mathcal{H}^{m-1}}^2 \, d\tau \\ &\leq C_1 Q(t) \int_0^t (\|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|u\|_{\mathcal{H}^m}^2) \, d\tau \end{aligned} \quad (3.26)$$

and

$$\int_0^t \|u \cdot [\mathcal{Z}^\alpha, \nabla] u\|^2 \, d\tau \leq \sum_{|\beta| \leq m-1} \|u\|_{L_{x,t}^\infty}^2 \int_0^t \|\mathcal{Z}^\beta \nabla u\|^2 \, d\tau \leq C \|u\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla u\|_{\mathcal{H}^{m-1}}^2 \, d\tau. \quad (3.27)$$

The combination of (3.25)-(3.27) yields that

$$\int_0^t \int C_1^\alpha \cdot \mathcal{Z}^\alpha u \, dx d\tau \leq C_1 Q(t) \int_0^t (\|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|u\|_{\mathcal{H}^m}^2) \, d\tau. \quad (3.28)$$

To deal with the pressure term  $-\int_0^t \int \mathcal{Z}^\alpha \nabla p \cdot \mathcal{Z}^\alpha u \, dx d\tau$ . Since  $\mathcal{Z}^\alpha u \cdot n$  does not vanish on the boundary, we follow the idea as Masmoudi and Rousset [14] to split the pressure  $p$  into the Euler part and Navier-Stokes part. More precisely, the pressure  $p$  is split as  $p = p_1 + p_2$ , where  $p_1$  is the "Euler" part of the pressure which solves

$$\Delta p_1 = -\operatorname{div}(u \cdot \nabla u + \nabla d \cdot \Delta d), \quad x \in \Omega, \quad \partial_n p_1 = -(u \cdot \nabla u + \nabla d \cdot \Delta d) \cdot n, \quad x \in \partial\Omega,$$

and  $p_2$  is the "Navier-Stokes part" which solves

$$\Delta p_2 = 0, \quad x \in \Omega, \quad \partial_n p_2 = (\varepsilon \Delta u - u_t) \cdot n, \quad x \in \Omega,$$

which, together with the boundary condition (1.3), gives directly

$$\Delta p_1 = -\operatorname{div}(u \cdot \nabla u + \nabla d \cdot \Delta d), \quad x \in \Omega, \quad \partial_n p_1 = -u \cdot \nabla u \cdot n, \quad x \in \partial\Omega, \quad (3.29)$$

and

$$\Delta p_2 = 0, \quad x \in \Omega, \quad \partial_n p_2 = \varepsilon \Delta u \cdot n, \quad x \in \Omega. \quad (3.30)$$

Then, it is easy to deduce that

$$-\int_0^t \int \mathcal{Z}^\alpha \nabla p \cdot \mathcal{Z}^\alpha u \, dx d\tau \leq \int_0^t \|\nabla^2 p_1\|_{\mathcal{H}^{m-1}} \|u\|_{\mathcal{H}^m} \, d\tau + \left| \int_0^t \int \mathcal{Z}^\alpha \nabla p_2 \cdot \mathcal{Z}^\alpha u \, dx d\tau \right|. \quad (3.31)$$

For the last term, we have

$$\begin{aligned}
 & \left| \int_0^t \int \mathcal{Z}^\alpha \nabla p_2 \cdot \mathcal{Z}^\alpha u dx d\tau \right| \\
 &= \left| \int_0^t \int \nabla \mathcal{Z}^\alpha p_2 \cdot \mathcal{Z}^\alpha u dx d\tau + \int_0^t \int [\mathcal{Z}^\alpha, \nabla] p_2 \cdot \mathcal{Z}^\alpha u dx d\tau \right| \\
 &\leq \left| \int_0^t \int \nabla \mathcal{Z}^\alpha p_2 \cdot \mathcal{Z}^\alpha u dx d\tau \right| + C_{m+1} \int_0^t \|\nabla p_2\|_{\mathcal{H}^{m-1}} \|u\|_{\mathcal{H}^m} d\tau.
 \end{aligned} \tag{3.32}$$

On the other hand, integrating by part and applying the divergence free of velocity, we find

$$\begin{aligned}
 & \left| \int_0^t \int \nabla \mathcal{Z}^\alpha p_2 \cdot \mathcal{Z}^\alpha u dx d\tau \right| \\
 &= \left| \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha p_2 \mathcal{Z}^\alpha u \cdot n d\sigma d\tau + \int_0^t \int \mathcal{Z}^\alpha p_2 [\mathcal{Z}^\alpha, \nabla \cdot] u dx d\tau \right| \\
 &\leq \left| \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha p_2 \mathcal{Z}^\alpha u \cdot n d\sigma d\tau \right| + \int_0^t \|\nabla p_2\|_{\mathcal{H}^{m-1}} \|\nabla u\|_{\mathcal{H}^{m-1}} d\tau.
 \end{aligned} \tag{3.33}$$

First, we deal with the boundary term when  $\alpha_{13} = 0$  (for  $\alpha_{13} \neq 0$ ,  $\mathcal{Z}^\alpha u = 0$  on the boundary) on the right hand side of (3.33). If  $|\alpha_0| = |\alpha|$ , it is easy to deduce that

$$\int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha p_2 \mathcal{Z}^\alpha u \cdot n d\sigma d\tau = 0. \tag{3.34}$$

If  $|\alpha_1| \geq 1$ , we integrating by parts along the boundary to obtain

$$\left| \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha p_2 \mathcal{Z}^\alpha u \cdot n d\sigma d\tau \right| \leq C_2 \int_0^t |\partial_t^{\alpha_0} Z_y^\beta p_2|_{L^2(\partial\Omega)} |\mathcal{Z}^\alpha u \cdot n|_{H^1(\partial\Omega)} d\tau, \tag{3.35}$$

where  $|\beta| = m - 1 - \alpha_0$ . Applying the trace Theorem in Proposition 2.3, we find

$$|\partial_t^{\alpha_0} Z_y^\beta p_2|_{L^2(\partial\Omega)} \leq C \|\nabla \partial_t^{\alpha_0} Z_y^\beta p_2\| + \|\partial_t^{\alpha_0} Z_y^\beta p_2\|_1 \leq C_1 \|\nabla p_2\|_{\mathcal{H}^{m-1}}, \tag{3.36}$$

and

$$|\mathcal{Z}^\alpha u \cdot n|_{H^1(\partial\Omega)} \leq C_{m+2} (\|\nabla u\|_{\mathcal{H}^m} + \|u\|_{\mathcal{H}^m}). \tag{3.37}$$

Substituting (3.34)-(3.37) into (3.33), we find

$$\left| \int_0^t \int \mathcal{Z}^\alpha \nabla p_2 \cdot \mathcal{Z}^\alpha u dx d\tau \right| \leq C_{m+2} \int_0^t \|\nabla p_2\|_{\mathcal{H}^{m-1}} (\|\nabla u\|_{\mathcal{H}^m} + \|u\|_{\mathcal{H}^m}) d\tau,$$

which, together with (3.31), reads

$$\begin{aligned}
 & \left| \int_0^t \int \mathcal{Z}^\alpha \nabla p \cdot \mathcal{Z}^\alpha u dx d\tau \right| \\
 &\leq C_{m+2} \int_0^t \|\nabla^2 p_1\|_{\mathcal{H}^{m-1}} \|u\|_{\mathcal{H}^m} d\tau + C_{m+2} \int_0^t \|\nabla p_2\|_{\mathcal{H}^{m-1}} (\|\nabla u\|_{\mathcal{H}^m} + \|u\|_{\mathcal{H}^m}) d\tau.
 \end{aligned} \tag{3.38}$$

We apply the Proposition 2.2 to deduce that

$$\int_0^t \|\mathcal{Z}^\alpha (\nabla d \cdot \Delta d)\|^2 d\tau \leq \|\nabla d\|_{L_{x,t}^\infty}^2 \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau + \|\Delta d\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau,$$

which implies that

$$\begin{aligned} & \left| - \int_0^t \int \mathcal{Z}^\alpha (\nabla d \cdot \Delta d) \cdot \mathcal{Z}^\alpha u \, dx d\tau \right| \\ & \leq \delta_1 \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau + C_{\delta_1} (\|\nabla d\|_{L_{x,t}^\infty}^2 + \|\Delta d\|_{L_{x,t}^\infty}^2) \int_0^t (\|u\|_{\mathcal{H}^m}^2 + \|\nabla d\|_{\mathcal{H}^m}^2) d\tau. \end{aligned} \quad (3.39)$$

Substituting (3.28), (3.38) and (3.39) into (3.24), it is easy to deduce

$$\begin{aligned} & \frac{1}{2} \int |\mathcal{Z}^\alpha u|^2 dx + \frac{3\varepsilon}{4} \int_0^t \int |\nabla \mathcal{Z}^\alpha u|^2 dx d\tau \\ & \leq \frac{1}{2} \int |\mathcal{Z}^\alpha u_0|^2 dx + \delta_1 \varepsilon \int_0^t \|\nabla u\|_{\mathcal{H}^m}^2 d\tau + \delta_1 \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau + \delta \varepsilon^2 \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau \\ & \quad + C_\delta (1 + Q(t)) \int_0^t N_m(\tau) d\tau + C_{m+2} C_\delta \int_0^t (\|\nabla^2 p_1\|_{\mathcal{H}^{m-1}} \|u\|_{\mathcal{H}^m} + \varepsilon^{-1} \|\nabla p_2\|_{\mathcal{H}^{m-1}}^2) d\tau. \end{aligned} \quad (3.40)$$

Applying the operator  $\mathcal{Z}^\alpha \nabla (|\alpha_0| + |\alpha_1| = m)$  to the equation (1.1)<sub>2</sub>, we find

$$\mathcal{Z}^\alpha \nabla d_t - \mathcal{Z}^\alpha \nabla \Delta d = -\mathcal{Z}^\alpha \nabla (u \cdot \nabla d) + \mathcal{Z}^\alpha \nabla (|\nabla d|^2 d). \quad (3.41)$$

Multiplying (3.41) by  $\mathcal{Z}^\alpha \nabla d$ , it is easy to deduce that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |\mathcal{Z}^\alpha \nabla d|^2 dx - \int \mathcal{Z}^\alpha \nabla \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx \\ & = - \int \mathcal{Z}^\alpha \nabla (u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \, dx + \int \mathcal{Z}^\alpha \nabla (|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx. \end{aligned} \quad (3.42)$$

Integrating by part, we find

$$\begin{aligned} & - \int \mathcal{Z}^\alpha \nabla \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx \\ & = - \int \nabla \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx - \int [\mathcal{Z}^\alpha, \nabla] \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx \\ & = - \int_{\partial\Omega} \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma + \int \mathcal{Z}^\alpha \Delta d \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx - \int [\mathcal{Z}^\alpha, \nabla] \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx \\ & = - \int_{\partial\Omega} \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma + \int |\operatorname{div}(\mathcal{Z}^\alpha \nabla d)|^2 dx + \int [\mathcal{Z}^\alpha, \operatorname{div}] \nabla d \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx \\ & \quad - \int [\mathcal{Z}^\alpha, \nabla] \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx. \end{aligned}$$

This, together with (3.42), reads

$$\begin{aligned} & \frac{1}{2} \int |\mathcal{Z}^\alpha \nabla d(t)|^2 dx + \int_0^t \int |\operatorname{div}(\mathcal{Z}^\alpha \nabla d)|^2 dx d\tau \\ & = \frac{1}{2} \int |\mathcal{Z}^\alpha \nabla d_0|^2 dx - \int_0^t \int \mathcal{Z}^\alpha \nabla (u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\ & \quad + \int_0^t \int \mathcal{Z}^\alpha \nabla (|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau - \int_0^t \int [\mathcal{Z}^\alpha, \operatorname{div}] \nabla d \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau \\ & \quad + \int_0^t \int [\mathcal{Z}^\alpha, \nabla] \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau + \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau. \end{aligned} \quad (3.43)$$

Integrating by part, it is easy to deduce

$$\begin{aligned}
& - \int_0^t \int \mathcal{Z}^\alpha \nabla (u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\
& = - \int_0^t \int \nabla \mathcal{Z}^\alpha (u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau - \int_0^t \int [\mathcal{Z}^\alpha, \nabla] (u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\
& = - \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha (u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau - \int_0^t \int \mathcal{Z}^\alpha (u \cdot \nabla d) \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau \\
& \quad - \int_0^t \int [\mathcal{Z}^\alpha, \nabla] (u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau.
\end{aligned} \tag{3.44}$$

To estimate the boundary term on the right hand side of (3.44). If  $|\alpha_0| = m$  or  $|\alpha_{13}| \geq 1$ , we obtain

$$- \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha (u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau = 0.$$

For  $|\beta| = m - 1 - \alpha_0$  ( $|\alpha_0| \leq m - 1$ ), we integrating by part along the boundary to deduce that

$$- \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha (u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau \leq \int_0^t |\partial_t^{\alpha_0} Z_y^\beta (u \cdot \nabla d)|_{L^2(\partial\Omega)} |\mathcal{Z}^\alpha \nabla d \cdot n|_{H^1(\partial\Omega)} d\tau. \tag{3.45}$$

Applying the trace theorem in Proposition 2.3 and Proposition 2.2, one arrives at

$$\begin{aligned}
& \int_0^t |\partial_t^{\alpha_0} Z_y^\beta (u \cdot \nabla d)|_{L^2(\partial\Omega)}^2 d\tau \\
& \leq C \int_0^t (\|\nabla \partial_t^{\alpha_0} (u \cdot \nabla d)\|_{m-1-\alpha_0}^2 + \|\partial_t^{\alpha_0} (u \cdot \nabla d)\|_{m-1-\alpha_0}^2) d\tau \\
& \leq C \int_0^t (\|\nabla (u \cdot \nabla d)\|_{\mathcal{H}^{m-1}}^2 + \|u \cdot \nabla d\|_{\mathcal{H}^{m-1}}^2) d\tau \\
& \leq C Q(t) \int_0^t (\|(u, \nabla d)\|_{\mathcal{H}^{m-1}}^2 + \|\nabla (u, \nabla d)\|_{\mathcal{H}^{m-1}}^2) d\tau.
\end{aligned} \tag{3.46}$$

With the help of boundary condition (1.3) and trace theorem in Proposition 2.3, we find

$$\int_0^t |\mathcal{Z}^\alpha \nabla d \cdot n|_{H^1(\partial\Omega)}^2 d\tau \leq C_{m+2} \int_0^t (\|\nabla^2 d\|_{\mathcal{H}^m}^2 + \|\nabla d\|_{\mathcal{H}^m}^2) d\tau. \tag{3.47}$$

The combination of (3.44)-(3.48) and Young inequality, it is easy to deduce that

$$\begin{aligned}
& - \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha (u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau \\
& \leq \delta_1 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^m}^2 d\tau + C_{\delta_1} C_{m+2} (1 + Q(t)) \int_0^t (\|(u, \nabla d)\|_{\mathcal{H}^m}^2 + \|\nabla (u, \nabla d)\|_{\mathcal{H}^{m-1}}^2) d\tau.
\end{aligned} \tag{3.48}$$

Applying the Young inequality and the Proposition 2.2, one arrives at

$$\begin{aligned}
& - \int_0^t \int \mathcal{Z}^\alpha (u \cdot \nabla d) \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau \\
& \leq \delta_1 \int_0^t \|\operatorname{div}(\mathcal{Z}^\alpha \nabla d)\|^2 d\tau + C_{\delta_1} \|u\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau + C_{\delta_1} \|\nabla d\|_{L_{x,t}^\infty}^2 \int_0^t \|u\|_{\mathcal{H}^m}^2 d\tau \\
& \leq \delta_1 \int_0^t \|\operatorname{div}(\mathcal{Z}^\alpha \nabla d)\|^2 d\tau + C_{\delta_1} C_1 Q(t) \int_0^t (\|u\|_{\mathcal{H}^m}^2 + \|\nabla d\|_{\mathcal{H}^m}^2) d\tau
\end{aligned} \tag{3.49}$$



and

$$\begin{aligned}
 & - \int_0^t \int [\mathcal{Z}^\alpha, \nabla](u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\
 & \leq \sum_{|\beta| \leq m-1} \int_0^t \|\mathcal{Z}^\beta (\nabla u \cdot \nabla d + u \cdot \nabla^2 d)\| \|\mathcal{Z}^\alpha \nabla d\| \, dx \\
 & \leq C \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau + C(\|\nabla u\|_{L_{x,t}^\infty}^2 + \|\nabla d\|_{L_{x,t}^\infty}^2) \int_0^t (\|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|\nabla d\|_{\mathcal{H}^{m-1}}^2) d\tau \\
 & \quad + C(\|u\|_{L_{x,t}^\infty}^2 + \|\nabla^2 d\|_{L_{x,t}^\infty}^2) \int_0^t (\|u\|_{\mathcal{H}^{m-1}}^2 + \|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2) d\tau \\
 & \leq C(1 + Q(t)) \int_0^t (\|u\|_{\mathcal{H}^{m-1}}^2 + \|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|\nabla d\|_{\mathcal{H}^m}^2 + \|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2) d\tau.
 \end{aligned} \tag{3.50}$$

Substituting (3.48)-(3.50) into (3.44), we obtain

$$\begin{aligned}
 & - \int_0^t \int \mathcal{Z}^\alpha \nabla(u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\
 & \leq \delta_1 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^m}^2 d\tau + C_{\delta_1} C_1 (1 + Q(t)) \int_0^t (\|(u, \nabla d)\|_{\mathcal{H}^m}^2 + \|\nabla(u, \nabla d)\|_{\mathcal{H}^{m-1}}^2) d\tau.
 \end{aligned} \tag{3.51}$$

On the other hand, it is complicated to deal with the term  $\int_0^t \int \mathcal{Z}^\alpha \nabla(|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau$ . Indeed, by integrating by part, one arrives at

$$\begin{aligned}
 & \int_0^t \int \mathcal{Z}^\alpha \nabla(|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\
 & = \int_0^t \int \nabla \mathcal{Z}^\alpha(|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau + \int_0^t \int [\mathcal{Z}^\alpha, \nabla](|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\
 & = - \int_0^t \int \mathcal{Z}^\alpha(|\nabla d|^2 d) \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau + \int_0^t \int [\mathcal{Z}^\alpha, \nabla](|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\
 & \quad + \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha(|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau.
 \end{aligned} \tag{3.52}$$

It is easy to deduce that

$$\begin{aligned}
 & - \int_0^t \int \mathcal{Z}^\alpha(|\nabla d|^2 d) \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau \\
 & = - \sum_{|\beta| \geq 1} \int_0^t \int \mathcal{Z}^\gamma(|\nabla d|^2) \mathcal{Z}^\beta d \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau - \int_0^t \int \mathcal{Z}^\alpha(|\nabla d|^2) d \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau.
 \end{aligned} \tag{3.53}$$

By virtue of the Proposition 2.2, we obtain

$$\begin{aligned}
 & \sum_{|\beta| \geq 1} \int_0^t \|\mathcal{Z}^\gamma(|\nabla d|^2) \mathcal{Z}^\beta d\|^2 d\tau \\
 & \leq \|\mathcal{Z}d\|_{L_{x,t}^\infty}^2 \int_0^t \| |\nabla d|^2 \|_{\mathcal{H}^{m-1}}^2 d\tau + \| |\nabla d|^2 \|_{L_{x,t}^\infty}^2 \int_0^t \|\mathcal{Z}d\|_{\mathcal{H}^{m-1}}^2 d\tau \\
 & \leq \|\mathcal{Z}d\|_{L_{x,t}^\infty}^2 \|\nabla d\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau + \|\nabla d\|_{L_{x,t}^\infty}^4 \int_0^t \|\mathcal{Z}d\|_{\mathcal{H}^{m-1}}^2 d\tau \\
 & \leq \|\nabla d\|_{L_{x,t}^\infty}^4 \int_0^t \|\partial_t d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_1(1 + P(Q(t))) \int_0^t N_m(\tau) d\tau.
 \end{aligned} \tag{3.54}$$

By virtue of equation (1.1)<sub>2</sub>, we find

$$\begin{aligned} \int_0^t \|\partial_t d\|_{\mathcal{H}^{m-1}}^2 d\tau &\leq \|u\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau + \|\nabla d\|_{L_{x,t}^\infty}^2 \int_0^t \|u\|_{\mathcal{H}^{m-1}}^2 d\tau \\ &\quad + \int_0^t \|\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + \int_0^t \| |\nabla d|^2 d \|_{\mathcal{H}^{m-1}}^2 d\tau. \end{aligned} \quad (3.55)$$

By virtue of the Proposition 2.2, we obtain

$$\begin{aligned} \int_0^t \| |\nabla d|^2 d \|_{\mathcal{H}^{m-1}}^2 d\tau &\leq \sum_{|\gamma| \geq 1, |\beta| + |\gamma| \leq m-1} \int_0^t \|\mathcal{Z}^\beta (|\nabla d|^2) \mathcal{Z}^\gamma d\|^2 d\tau + \int_0^t \| |\nabla d|^2 \|_{\mathcal{H}^{m-1}}^2 d\tau \\ &\leq \|\mathcal{Z} d\|_{L_{x,t}^\infty}^2 \|\nabla d\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-2}}^2 d\tau + \|\nabla d\|_{L_{x,t}^\infty}^4 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-2}}^2 d\tau \\ &\quad + \|\nabla d\|_{L_{x,t}^\infty}^4 \int_0^t \|\partial_t d\|_{\mathcal{H}^{m-2}}^2 d\tau + \|\nabla d\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau. \end{aligned} \quad (3.56)$$

Substituting (3.56) into (3.55), we obtain

$$\begin{aligned} \int_0^t \|d_t\|_{\mathcal{H}^{m-1}}^2 d\tau &\leq \|\nabla d\|_{L_{x,t}^\infty}^4 \int_0^t \|\partial_t d\|_{\mathcal{H}^{m-2}}^2 d\tau + \int_0^t \|\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \\ &\quad + C_1(1 + P(Q(t))) \int_0^t (\|u\|_{\mathcal{H}^{m-1}}^2 + \|\nabla d\|_{\mathcal{H}^{m-1}}^2) d\tau. \end{aligned} \quad (3.57)$$

On the other hand, it is easy to deduce that

$$\int_0^t \|d_t\|^2 d\tau \leq \int_0^t \|\Delta d\|^2 d\tau + (1 + \|\nabla d\|_{L_{x,t}^\infty}^2) \int_0^t (\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) d\tau. \quad (3.58)$$

The combination of (3.57) and (3.58) yields directly

$$\begin{aligned} \int_0^t \|d_t\|_{\mathcal{H}^{m-1}}^2 d\tau &\leq C_1(1 + P(Q(t))) \int_0^t \|\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \\ &\quad + C_1(1 + P(Q(t))) \int_0^t (\|u\|_{\mathcal{H}^{m-1}}^2 + \|\nabla d\|_{\mathcal{H}^{m-1}}^2) d\tau, \end{aligned} \quad (3.59)$$

which, together with (3.54), gives directly

$$\begin{aligned} &\left| - \sum_{|\beta| \geq 1} \int_0^t \int \mathcal{Z}^\gamma (|\nabla d|^2) \mathcal{Z}^\beta d \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau \right| \\ &\leq \delta_1 \int_0^t \|\operatorname{div}(\mathcal{Z}^\alpha \nabla d)\| d\tau + C_{\delta_1} C_1(1 + P(Q(t))) \int_0^t N_m(\tau) d\tau. \end{aligned} \quad (3.60)$$

In view of the Proposition 2.2 and Cauchy inequality, we obtain

$$\begin{aligned} &\left| - \int_0^t \int \mathcal{Z}^\alpha (|\nabla d|^2) d \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau \right| \\ &\leq \delta_1 \int_0^t \|\operatorname{div}(\mathcal{Z}^\alpha \nabla d)\| d\tau + C_{\delta_1} \|\nabla d\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau. \end{aligned} \quad (3.61)$$

Then combination of (3.60) and (3.61) yields immediately

$$\begin{aligned} & - \int_0^t \int \mathcal{Z}^\alpha (|\nabla d|^2 d) \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau \\ & \leq \delta_1 \int_0^t \|\operatorname{div}(\mathcal{Z}^\alpha \nabla d)\| d\tau + C_1 C_{\delta_1} (1 + P(Q(t))) \int_0^t N_m(\tau) d\tau. \end{aligned} \quad (3.62)$$

On the other hand, we find that

$$\begin{aligned} & \int_0^t \int [\mathcal{Z}^\alpha, \nabla] (|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\ & = \sum_{|\beta| \leq m-1} \int_0^t \int d \cdot \mathcal{Z}^\beta (\nabla d \cdot \nabla^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\ & \quad + \sum_{\substack{|\beta| \geq 1 \\ |\beta| + |\gamma| \leq m-1}} \int_0^t \int \mathcal{Z}^\beta d \cdot \mathcal{Z}^\gamma (\nabla d \cdot \nabla^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\ & \quad + \sum_{|\beta| + |\gamma| \leq m-1} \int_0^t \int \mathcal{Z}^\beta (|\nabla d|^2) \mathcal{Z}^\gamma \nabla d \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\ & = I_1 + I_2 + I_3. \end{aligned} \quad (3.63)$$

In view of the Proposition 2.2, we find

$$I_1 \leq \|\nabla d\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2 d\tau + \|\nabla^2 d\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau + \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau. \quad (3.64)$$

and

$$\begin{aligned} I_2 & \leq C \|\nabla d\|_{W_{x,t}^{1,\infty}}^4 \int_0^t \|\mathcal{Z} d\|_{\mathcal{H}^{m-2}}^2 d\tau + C \|\mathcal{Z} d\|_{L_{x,t}^\infty}^2 \|\nabla d\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^{m-2}}^2 d\tau \\ & \quad + C \|\mathcal{Z} d\|_{L_{x,t}^\infty}^2 \|\nabla^2 d\|_{L^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-2}}^2 d\tau + C \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau \\ & \leq C_1 (1 + P(Q(t))) \int_0^t N_m(\tau) d\tau, \end{aligned} \quad (3.65)$$

where we have used the estimate (3.59). Similarly, it is easy to deduce that

$$I_3 \leq C \|\nabla d\|_{L_{x,t}^\infty}^4 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau + C \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau. \quad (3.66)$$

Substituting (3.64)-(3.66) into (3.63), one arrives at

$$\int_0^t \int [\mathcal{Z}^\alpha, \nabla] (|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \leq C_1 (1 + P(Q(t))) \int_0^t N_m(\tau) d\tau. \quad (3.67)$$

Deal with the boundary term on the right hand side of (3.52). If  $|\alpha_0| = m$  or  $|\alpha_{13}| \geq 1$ , we obtain

$$\int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha (|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau = 0. \quad (3.68)$$

Indeed, it is easy to deduce that for  $|\beta| = m - 1 - \alpha_0$

$$\int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha (|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau \leq \int_0^t |\partial_t^{\alpha_0} \mathcal{Z}_y^\beta (|\nabla d|^2 d)|_{L^2(\partial\Omega)} |\mathcal{Z}^\alpha \nabla d \cdot n|_{H^1(\partial\Omega)} d\tau. \quad (3.69)$$

By virtue of the trace theorem in Proposition 2.3, we find for  $|\beta| = m - 1 - \alpha_0$

$$\begin{aligned} & |\partial_t^{\alpha_0} Z_y^\beta (|\nabla d|^2 d)|_{L^2(\partial\Omega)}^2 \\ & \leq C \|\nabla \partial_t^{\alpha_0} (|\nabla d|^2 d)\|_{m-1-\alpha_0} \|\partial_t^{\alpha_0} (|\nabla d|^2 d)\|_{m-1-\alpha_0} + C \|\partial_t^{\alpha_0} (|\nabla d|^2 d)\|_{m-1-\alpha_0}^2 \\ & \leq C \|\nabla (|\nabla d|^2 d)\|_{\mathcal{H}^{m-1}} \|\nabla d\|_{\mathcal{H}^{m-1}}^2 + C \|\nabla d\|_{\mathcal{H}^{m-1}}^2, \end{aligned} \quad (3.70)$$

and

$$|\mathcal{Z}^\alpha \nabla d \cdot n|_{H^1(\partial\Omega)}^2 \leq C_{m+2} (\|\nabla^2 d\|_{\mathcal{H}^m}^2 + \|\nabla d\|_{\mathcal{H}^m}^2). \quad (3.71)$$

On the other hand, we obtain just following the idea as (3.53) and (3.63) that

$$\int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau \leq C(1 + P(Q(t))) \int_0^t N_m(\tau) d\tau \quad (3.72)$$

and

$$\int_0^t \|\nabla (|\nabla d|^2 d)\|_{\mathcal{H}^{m-1}}^2 d\tau \leq C(1 + P(Q(t))) \int_0^t N_m(\tau) d\tau. \quad (3.73)$$

The combination of (3.69)-(3.73) gives directly

$$\begin{aligned} & \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha (|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau \\ & \leq \delta_1 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^m}^2 d\tau + C_{\delta_1} C_{m+2} (1 + P(Q(t))) \int_0^t N_m(\tau) d\tau. \end{aligned} \quad (3.74)$$

Substituting (3.60), (3.67) and (3.74) into (3.52), we attains

$$\begin{aligned} & \int_0^t \int \mathcal{Z}^\alpha \nabla (|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau \\ & \leq \delta_1 \int_0^t \|\operatorname{div}(\mathcal{Z}^\alpha \nabla d)\|^2 d\tau + \delta_1 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^m}^2 d\tau + C_{\delta_1} C_{m+2} (1 + P(Q(t))) \int_0^t N_m(\tau) d\tau. \end{aligned} \quad (3.75)$$

In view of the Cauchy inequality, it is easy to deduce that

$$- \int_0^t \int [\mathcal{Z}^\alpha, \operatorname{div}] \nabla d \cdot \operatorname{div}(\mathcal{Z}^\alpha \nabla d) \, dx d\tau \leq \delta_1 \int_0^t \|\operatorname{div}(\mathcal{Z}^\alpha \nabla d)\|^2 d\tau + C_{\delta_1} \int_0^t \|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2 d\tau, \quad (3.76)$$

and

$$\int_0^t \int [\mathcal{Z}^\alpha, \nabla] \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx \leq \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau. \quad (3.77)$$

Finally, we deal with the boundary term on the right hand side of (3.43). If  $|\alpha_0| = m$  or  $|\alpha_{13}| \geq 1$ , we obtain

$$\int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau = 0. \quad (3.78)$$

On the other hand, integrating by part along the boundary, we have for  $|\beta| = m - 1 - \alpha_0$

$$\int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma d\tau \leq C \int_0^t |\partial_t^{\alpha_0} Z_y^\beta \Delta d|_{L^2(\partial\Omega)} |\mathcal{Z}^\alpha \nabla d \cdot n|_{H^1(\partial\Omega)} d\tau. \quad (3.79)$$

By virtue of the trace theorem in Proposition 2.3, one arrives at

$$\begin{aligned} |\partial_t^{\alpha_0} Z_y^\beta \Delta d|_{L^2(\partial\Omega)} & \leq C (\|\nabla \partial_t^{\alpha_0} Z_y^\beta \Delta d\|^{\frac{1}{2}} + \|\partial_t^{\alpha_0} Z_y^\beta \Delta d\|^{\frac{1}{2}}) \|\partial_t^{\alpha_0} Z_y^\beta \Delta d\|^{\frac{1}{2}} \\ & \leq C (\|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^{\frac{1}{2}} + \|\Delta d\|_{\mathcal{H}^{m-1}}^{\frac{1}{2}}) \|\Delta d\|_{\mathcal{H}^{m-1}}^{\frac{1}{2}}. \end{aligned} \quad (3.80)$$

Similarly, with the help of boundary condition (1.3) and trace theorem, one arrives at

$$|\mathcal{Z}^\alpha \nabla d \cdot n|_{H^1(\partial\Omega)} \leq C_{m+2}(\|\nabla^2 d\|_{\mathcal{H}^m}^{\frac{1}{2}} + \|\nabla d\|_{\mathcal{H}^m}^{\frac{1}{2}})\|\nabla d\|_{\mathcal{H}^m}^{\frac{1}{2}}. \quad (3.81)$$

Substituting (3.80) and (3.81) into (3.79) and applying the Young inequality, we find

$$\begin{aligned} & \int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \cdot n d\sigma d\tau \\ & \leq \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + \delta_1 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^m}^2 d\tau + C_{\delta, \delta_1} C_{m+2} \int_0^t (\|\nabla d\|_{\mathcal{H}^m}^2 + \|\Delta d\|_{\mathcal{H}^{m-1}}^2) d\tau. \end{aligned} \quad (3.82)$$

Substituting (3.48), (3.51), (3.75)-(3.77) and (3.73) into (3.43) and choosing  $\delta_1$  small enough, one arrives at

$$\begin{aligned} & \frac{1}{2} \int |\mathcal{Z}^\alpha \nabla d(t)|^2 dx + \frac{3}{4} \int_0^t \int |\mathcal{Z}^\alpha \Delta d|^2 dx d\tau \\ & \leq \frac{1}{2} \int |\mathcal{Z}^\alpha \nabla d_0|^2 dx + \delta_2 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^m}^2 d\tau + \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \\ & \quad + C_{\delta, \delta_2} C_{m+2} (1 + P(Q(t))) \int_0^t N_m(\tau) d\tau. \end{aligned} \quad (3.83)$$

In view of the standard elliptic regularity results with Neumann boundary condition, we get that

$$\begin{aligned} \|\nabla^2 d\|_{\mathcal{H}^m}^2 &= \|\nabla^2 \partial_t^{\alpha_0} d\|_{m-\alpha_0}^2 \\ &\leq C_{m+2} (\|\nabla \partial_t^{\alpha_0} d\|^2 + \|\Delta \partial_t^{\alpha_0} d\|_{m-\alpha_0}^2) \\ &\leq C_{m+2} (\|\nabla d\|_{\mathcal{H}^m}^2 + \|\Delta d\|_{\mathcal{H}^m}^2). \end{aligned} \quad (3.84)$$

Hence, the combination of (3.40), (3.83) and (3.84) yields directly

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} \|(u, \nabla d)\|_{\mathcal{H}^m}^2 + \varepsilon \int_0^t \|\nabla u\|_{\mathcal{H}^m}^2 d\tau + \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau \\ & \leq C_{m+2} \left\{ \|(u_0, \nabla d_0)\|_{\mathcal{H}^m}^2 + \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + \delta \varepsilon^2 \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau \right. \\ & \quad \left. + \int_0^t (\|\nabla^2 p_1\|_{\mathcal{H}^{m-1}} \|u\|_{\mathcal{H}^m} + \varepsilon^{-1} \|\nabla p_2\|_{\mathcal{H}^{m-1}}^2) d\tau + C_\delta (1 + P(Q(t))) \int_0^t N_m(\tau) d\tau \right\}. \end{aligned}$$

Therefore, we complete the proof of lemma 3.3.  $\square$

### 3.2. Conormal Estimates for $\Delta d$

In this subsection, we shall get some uniform estimates for  $\Delta d$  in conormal Sobolev space.

**Lemma 3.4.** *For a smooth solution to (1.1) and (1.3), it holds that for  $\varepsilon \in (0, 1]$*

$$\sup_{0 \leq \tau \leq t} \|\Delta d(\tau)\|^2 + \int_0^t \|\nabla \Delta d\|^2 d\tau \leq \|\Delta d_0\|^2 + C(1 + Q(t)^2) \int_0^t N_1(\tau) d\tau. \quad (3.85)$$

*Proof.* Taking  $\nabla$  operator to the equation (1.1)<sub>2</sub>, one arrives at

$$\nabla d_t - \nabla \Delta d = -\nabla(u \cdot \nabla d) + \nabla(|\nabla d|^2 d). \quad (3.86)$$

Multiplying (3.86) by  $-\nabla \Delta d$  and integrating over  $\Omega$ , we find

$$\begin{aligned} & - \int \nabla d_t \cdot \nabla \Delta d \, dx + \int |\nabla \Delta d|^2 dx \\ & = \int \nabla(u \cdot \nabla d) \cdot \nabla \Delta d \, dx - \int \nabla(|\nabla d|^2 d) \cdot \nabla \Delta d \, dx. \end{aligned} \quad (3.87)$$

By integrating by parts and applying the Neumann boundary condition (1.3), we get

$$- \int \nabla d_t \cdot \nabla \Delta d \, dx = - \int_{\partial\Omega} n \cdot \nabla d_t \cdot \Delta d \, d\sigma + \int \Delta d_t \cdot \Delta d \, dx = \frac{1}{2} \frac{d}{dt} \int |\Delta d|^2 dx. \quad (3.88)$$

In view of the Cauchy inequality, we obtain

$$\begin{aligned} & \int \nabla(u \cdot \nabla d) \cdot \nabla \Delta d \, dx \leq \delta \|\nabla \Delta d\|^2 + C_\delta \|u\|_{W^{1,\infty}}^2 (\|\nabla d\|^2 + \|\nabla^2 d\|^2), \\ & - \int \nabla(|\nabla d|^2 d) \cdot \nabla \Delta d \, dx \leq \delta \|\nabla \Delta d\|^2 + \|\nabla d\|_{L^\infty}^4 \|\nabla d\|^2 + \|\nabla d\|^2 \|\nabla^2 d\|^2. \end{aligned} \quad (3.89)$$

Substituting (3.89) into (3.88), choosing  $\delta$  small enough and integrating over  $[0, t]$ , one attains

$$\begin{aligned} & \frac{1}{2} \int |\Delta d|^2(t) dx + \frac{3}{4} \int |\nabla \Delta d|^2 dx \\ & \leq \int |\Delta d_0|^2 dx + C(\|u\|_{W^{1,\infty}}^2 + \|\nabla d\|_{L^\infty}^4) \int_0^t (\|\nabla d\|^2 + \|\nabla^2 d\|^2) d\tau. \end{aligned}$$

Therefore, we complete the proof of lemma 3.4.  $\square$

Next, we can establish the following conormal estimates for the quantity  $\Delta d$ .

**Lemma 3.5.** *For  $m \geq 1$  and a smooth solution to (1.1) and (1.3), it holds that for  $\varepsilon \in (0, 1]$*

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} \|\Delta d(\tau)\|_{\mathcal{H}^{m-1}}^2 + \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \\ & \leq C_{m+2} \left\{ \|\Delta d_0\|_{\mathcal{H}^{m-1}}^2 + \delta \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau + C_\delta (1 + P(Q(t))) \int_0^t N_m(t) d\tau \right\}. \end{aligned} \quad (3.90)$$

*Proof.* The case for  $m = 1$  is already proved in Lemma 3.4. Assume that (3.90) is proved for  $k = m - 2$ . We shall prove that it holds for  $k = m - 1 \geq 1$ . For  $|\alpha| = m - 1$ , multiplying (3.86) by  $-\nabla \mathcal{Z}^\alpha \Delta d$ , we find

$$\begin{aligned} & - \int \mathcal{Z}^\alpha \nabla d_t \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx + \int \mathcal{Z}^\alpha \nabla \Delta d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx \\ & = \int \mathcal{Z}^\alpha \nabla(u \cdot \nabla d) \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx + \int \mathcal{Z}^\alpha \nabla(|\nabla d|^2 d) \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx. \end{aligned} \quad (3.91)$$

Integrating by part, it is easy to deduce that

$$\begin{aligned} & - \int \mathcal{Z}^\alpha \nabla d_t \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx \\ & = - \int_{\partial\Omega} n \cdot \mathcal{Z}^\alpha \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, d\sigma + \int \nabla \cdot (\mathcal{Z}^\alpha \nabla d_t) \cdot \mathcal{Z}^\alpha \Delta d \, dx \\ & = - \int_{\partial\Omega} n \cdot \mathcal{Z}^\alpha \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, d\sigma + \frac{1}{2} \frac{d}{dt} \int |\mathcal{Z}^\alpha \Delta d|^2 dx - \int [\mathcal{Z}^\alpha, \nabla \cdot] \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, dx. \end{aligned} \quad (3.92)$$

It is easy to deduce that

$$\int \mathcal{Z}^\alpha \nabla \Delta d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx = \int |\nabla \mathcal{Z}^\alpha \Delta d|^2 dx + \int [\mathcal{Z}^\alpha, \nabla] \Delta d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx. \quad (3.93)$$

Substituting (3.92) and (3.93) into (3.91) and integrating over  $[0, t]$ , one attains that

$$\begin{aligned} & \frac{1}{2} \int |\mathcal{Z}^\alpha \Delta d(t)|^2 dx + \int_0^t \int |\nabla \mathcal{Z}^\alpha \Delta d|^2 dx d\tau \\ &= \frac{1}{2} \int |\mathcal{Z}^\alpha \Delta d_0|^2 dx + \int_0^t \int_{\partial\Omega} n \cdot \mathcal{Z}^\alpha \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, d\sigma d\tau + \int_0^t \int [\mathcal{Z}^\alpha, \nabla \cdot] \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, dx d\tau \\ & \quad - \int_0^t \int [\mathcal{Z}^\alpha, \nabla] \Delta d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau + \int_0^t \int \mathcal{Z}^\alpha \nabla (u \cdot \nabla d) \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau \\ & \quad + \int_0^t \int \mathcal{Z}^\alpha \nabla (|\nabla d|^2 d) \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau \\ &:= II_1 + II_2 + II_3 + II_4 + II_5 + II_6. \end{aligned} \quad (3.94)$$

To deal with the boundary term on the right hand side of (3.94). If  $|\alpha_0| = m - 1$ , then we have

$$\int_0^t \int_{\partial\Omega} n \cdot \mathcal{Z}^\alpha \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, d\sigma d\tau = 0. \quad (3.95)$$

On the other hand, it is easy to deduce that for  $|\alpha_0| \leq m - 2$

$$\int_0^t \int_{\partial\Omega} n \cdot \mathcal{Z}^\alpha \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, d\sigma d\tau \leq \int_0^t |n \cdot \mathcal{Z}^\alpha \nabla d_t|_{L^2(\partial\Omega)} |\mathcal{Z}^\alpha \Delta d|_{L^2(\partial\Omega)} d\tau, \quad (3.96)$$

The application of trace inequality in Proposition 2.3 and the boundary condition (1.3) implies

$$\begin{aligned} |\mathcal{Z}^\alpha \Delta d|_{L^2(\partial\Omega)} &= |\partial_t^{\alpha_0} \Delta d|_{H^{m-1-|\alpha_0|}(\partial\Omega)} \\ &\leq C \|\nabla \partial_t^{\alpha_0} \Delta d\|_{m-1-|\alpha_0|} + C \|\partial_t^{\alpha_0} \Delta d\|_{m-|\alpha_0|} \\ &\leq C \|\nabla \Delta d\|_{\mathcal{H}^{m-1}} + C \|\Delta d\|_{\mathcal{H}^m}, \end{aligned} \quad (3.97)$$

and

$$\begin{aligned} |n \cdot \mathcal{Z}^\alpha \nabla d_t|_{L^2(\partial\Omega)} &\leq C_m |\partial_t^{\alpha_0} \nabla d_t|_{H^{m-2-|\alpha_0|}(\partial\Omega)} \\ &\leq C_m \|\partial_t^{\alpha_0} \nabla^2 d_t\|_{m-2-|\alpha_0|} + C_m \|\partial_t^{\alpha_0} \nabla d_t\|_{m-1-|\alpha_0|} \\ &\leq C_m \|\nabla^2 d\|_{\mathcal{H}^{m-1}} + C_m \|\nabla d\|_{\mathcal{H}^m}. \end{aligned} \quad (3.98)$$

Substituting (3.97) and (3.98) into (3.96) and applying the Cauchy inequality, one attains

$$\begin{aligned} II_2 &\leq \delta_1 \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + \delta \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau \\ &\quad + C_m \left\{ C_{\delta, \delta_1} \int_0^t \|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_{\delta, \delta_1} \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau \right\}. \end{aligned} \quad (3.99)$$

By virtue of the Cauchy inequality, one arrives at

$$\begin{aligned} II_3 &\leq C \int_0^t \|\Delta d_t\|_{\mathcal{H}^{m-2}} \|\Delta d\|_{\mathcal{H}^{m-1}} d\tau \leq C \int_0^t \|\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau, \\ II_4 &\leq \delta_1 \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|^2 d\tau + C_{\delta_1} \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-2}}^2 d\tau. \end{aligned} \quad (3.100)$$

The application of Proposition 2.2 yields directly

$$\begin{aligned}
II_5 &= \int_0^t \int \mathcal{Z}^\alpha (\nabla u \cdot \nabla d) \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau + \int_0^t \int \mathcal{Z}^\alpha (u \cdot \nabla^2 d) \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau \\
&\leq \delta_1 \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|^2 d\tau + C_\delta \|\nabla u\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta \|\nabla d\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla u\|_{\mathcal{H}^{m-1}}^2 d\tau \\
&\quad + C_\delta \|u\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta \|\nabla^2 d\|_{L_{x,t}^\infty}^2 \int_0^t \|u\|_{\mathcal{H}^{m-1}}^2 d\tau \\
&\leq \delta_1 \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|^2 d\tau + C_{\delta_1} (1 + P(Q(t))) \int_0^t N_m(\tau) d\tau.
\end{aligned} \tag{3.101}$$

It is easy to deduce that

$$\begin{aligned}
II_6 &= \sum_{|\beta| \geq 1} \int_0^t \int \mathcal{Z}^\gamma (\nabla d \cdot \nabla^2 d) \cdot \mathcal{Z}^\beta d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau \\
&\quad + \int_0^t \int \mathcal{Z}^\alpha (\nabla d \cdot \nabla^2 d) \cdot d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau \\
&\quad + \sum_{|\beta|+|\gamma|=m-1} \int_0^t \int \mathcal{Z}^\gamma (|\nabla d|^2) \mathcal{Z}^\beta \nabla d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau \\
&= II_{61} + II_{62} + II_{63}.
\end{aligned} \tag{3.102}$$

By virtue of the Proposition 2.2 and Cauchy inequality, one arrives at

$$\begin{aligned}
II_{61} &\leq \delta \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|^2 d\tau + C_\delta \|\mathcal{Z} d\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla d \cdot \nabla^2 d\|_{\mathcal{H}^{m-2}}^2 d\tau \\
&\quad + C_\delta \|\nabla d \cdot \nabla^2 d\|_{L_{x,t}^\infty}^2 \int_0^t \|\mathcal{Z} d\|_{\mathcal{H}^{m-2}}^2 d\tau \\
&\leq \delta_1 \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|^2 d\tau + C_1 C_{\delta_1} (1 + P(Q(t))) \int_0^t N_m(\tau) d\tau.
\end{aligned} \tag{3.103}$$

Similarly, it is easy to deduce that

$$\begin{aligned}
II_{62} &\leq \delta \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|^2 d\tau + C_\delta \|\nabla d\|_{W_{x,t}^{1,\infty}}^2 \int_0^t (\|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2 + \|\nabla d\|_{\mathcal{H}^{m-1}}^2) d\tau, \\
II_{63} &\leq \delta \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|^2 d\tau + C_\delta \|\nabla d\|_{L_{x,t}^\infty}^4 \int_0^t (\|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2 + \|\nabla d\|_{\mathcal{H}^{m-1}}^2) d\tau.
\end{aligned} \tag{3.104}$$

Substituting (3.103) and (3.104) into (3.102), we obtain

$$II_6 \leq \delta \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|^2 d\tau + C_1 C_\delta (1 + P(Q(t))) \int_0^t N_m(\tau) d\tau. \tag{3.105}$$

Substituting (3.99)-(3.101) and (3.105) into (3.94) and choosing  $\delta_1$  small enough, we find

$$\begin{aligned}
&\frac{1}{2} \int |\mathcal{Z}^\alpha \Delta d(t)|^2 dx + \int_0^t \int |\nabla \mathcal{Z}^\alpha \Delta d|^2 dx d\tau \\
&\leq C_m \left\{ \frac{1}{2} \int |\mathcal{Z}^\alpha \Delta d_0|^2 dx + \delta \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau + C \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-2}}^2 d\tau \right\} \\
&\quad + C_m C_\delta [1 + Q(t)^2] \int_0^t N_m(\tau) d\tau.
\end{aligned}$$



By the induction assumption, one can eliminate the term  $\int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-2}}^2 d\tau$ . Therefore, we complete the proof of the Lemma 3.5.  $\square$

### 3.3. Normal Derivatives Estimates

We shall now provide an estimate for  $\|\nabla u\|_{\mathcal{H}^{m-1}}$ . Of course, the only difficulty is to estimate  $\|\chi \partial_n u\|_{\mathcal{H}^{m-1}}$ , where  $\chi$  is compactly supported in one of the  $\Omega_i$  and with value one in a vicinity of the boundary. Indeed, we have by definition of the norm that  $\|\chi \partial_{y_i} u\|_{\mathcal{H}^{m-1}} \leq C\|u\|_{\mathcal{H}^m}$ ,  $i = 1, 2$ . We shall thus use the local coordinates.

At first, thanks to the divergence free condition, which reads

$$\operatorname{div} u = \partial_n u \cdot n + (\Pi \partial_{y_1} u)^1 + (\Pi \partial_{y_2} u)^2 = 0, \quad (3.106)$$

to get that

$$\|\chi \partial_n u \cdot n\|_{\mathcal{H}^{m-1}} \leq C_m \|u\|_{\mathcal{H}^m}. \quad (3.107)$$

Then, it remains to estimate  $\|\chi \Pi(\partial_n u)\|_{\mathcal{H}^{m-1}}$ . We extend the smooth symmetric matrix  $A$  in (1.2) to be  $A(y, z) = A(y)$ . Let us set

$$\eta \triangleq \chi \{w \times n - \Pi(Bu)\}. \quad (3.108)$$

In view of the Navier-slip type boundary condition (1.3), we obviously have that  $\eta$  satisfies a homogeneous Dirichlet boundary condition on the boundary

$$\eta|_{\partial\Omega} = 0. \quad (3.109)$$

By virtue of  $w \times n = (\nabla u - (\nabla u)^t) \cdot n$ , then  $\eta$  can be written as

$$\eta = \chi \Pi \{ \partial_n u - \nabla(u \cdot n) + (\nabla n)^t u + Bu \}, \quad (3.110)$$

which yields

$$\|\chi \Pi(\partial_n u)\|_{\mathcal{H}^{m-1}} \leq C_{m+1} (\|\eta\|_{\mathcal{H}^{m-1}} + \|u\|_{\mathcal{H}^m}). \quad (3.111)$$

Then, we find

$$\|\nabla u\|_{\mathcal{H}^{m-1}} \leq C_{m+1} (\|\eta\|_{\mathcal{H}^{m-1}} + \|u\|_{\mathcal{H}^m}). \quad (3.112)$$

On the other hand, it is easy to deduce that

$$\|\eta\|_{\mathcal{H}^{m-1}} \leq C_{m+1} (\|\nabla u\|_{\mathcal{H}^{m-1}} + \|u\|_{\mathcal{H}^m}). \quad (3.113)$$

Now we shall establish the estimates for the equivalent quantity  $\eta$ .

**Lemma 3.6.** *For a smooth solution to (1.1) and (1.3), it holds that for  $\varepsilon \in (0, 1]$*

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} \|\eta(\tau)\|^2 + \varepsilon \int_0^t \|\nabla \eta\|^2 d\tau \\ & \leq \int |\eta_0|^2 dx + \int_0^t \|\nabla p\| \|\eta\| d\tau + \delta \varepsilon^2 \int_0^t \|\nabla^2 u\|^2 d\tau + \delta \int_0^t \|\nabla \Delta d\|^2 d\tau \\ & \quad + C_3 C_\delta (1 + Q(t)) \int_0^t N_1(\tau) d\tau. \end{aligned} \quad (3.114)$$

*Proof.* For  $w \triangleq \nabla \times u$ , notice that

$$\nabla \times ((u \cdot \nabla)u) = (u \cdot \nabla)w - (w \cdot \nabla)u + w \operatorname{div} u,$$

which, together with (3.3), reads

$$w_t + (u \cdot \nabla)w - \varepsilon \Delta w = w \cdot \nabla u - \nabla \times (\nabla d \cdot \Delta d) \quad (3.115)$$

Consequently, the system for  $\eta$  is

$$\eta_t + u \cdot \nabla \eta - \varepsilon \Delta \eta = \chi[F_1 \times n + \Pi(BF_2)] + \chi F_3 + F_4 + \varepsilon \Delta(\Pi B) \cdot u, \quad (3.116)$$

where

$$\begin{aligned} F_1 &= w \cdot \nabla u - \nabla \times (\nabla d \cdot \Delta d), \\ F_2 &= -\nabla p - \nabla d \cdot \Delta d, \\ F_3 &= -2\varepsilon \sum_{i=1}^3 \partial_i w \times \partial_i n - \varepsilon w \times \Delta n + \sum_{i=1}^3 u_i w \times \partial_i n \\ &\quad + u \cdot \nabla(\Pi B)u - 2\varepsilon \sum_{i=1}^3 \partial_i(\Pi B)\partial_i u, \\ F_4 &= (u \cdot \nabla \chi)[w \times n + \Pi(Bu)] - \varepsilon \Delta \chi[w \times n + \Pi(Bu)] \\ &\quad - 2\varepsilon \sum_{i=1}^3 \partial_i \chi \partial_i [w \times n + \Pi(Bu)]. \end{aligned}$$

Multiplying (3.116) by  $\eta$ , it is easy to deduce that

$$\frac{1}{2} \frac{d}{dt} \int |\eta|^2 dx + \varepsilon \int |\nabla \eta|^2 dx = \int F \cdot \eta dx + \varepsilon \int \Delta(\Pi B) \cdot u \cdot \eta dx. \quad (3.117)$$

It is easy to deduce that

$$\begin{aligned} \|\chi[F_1 \times n + \Pi(BF_2)]\| &\leq C_2(\|\nabla u\|_{L^\infty} \|\nabla u\| + \|\nabla d\|_{L^\infty} (\|\nabla \Delta d\| + \|\Delta d\|) + \|\nabla p\|), \\ \|\chi F_3\| &\leq \varepsilon \|\nabla^2 u\| + C_3(1 + \|u\|_{L^\infty})(\|u\| + \|\nabla u\|). \end{aligned} \quad (3.118)$$

Notice that the term  $F_4$  are supported away from the boundary, we can control all the derivatives by the  $\|\cdot\|_{\mathcal{H}^m}$ . Hence, we find

$$\|F_4\| \leq \varepsilon \|\nabla^2 u\| + C_3(1 + \|u\|_{L^\infty})\|u\|_{\mathcal{H}^1}. \quad (3.119)$$

Integrating by parts, it is easy to deduce that

$$\varepsilon \int \Delta(\Pi B) \cdot u \cdot \eta dx \leq \delta \varepsilon \int |\nabla \eta|^2 dx + C_\delta C_3(\|\nabla u\|^2 + \|u\|_{\mathcal{H}^1}^2). \quad (3.120)$$

Substituting (3.118)-(3.120) into (3.117) and integrating the resulting inequality over  $[0, t]$ , we have

$$\begin{aligned} &\frac{1}{2} \int |\eta|^2(t) dx + \varepsilon \int_0^t \int |\nabla \eta|^2 dx d\tau \\ &\leq \frac{1}{2} \int |\eta_0|^2 dx + \int_0^t \|\nabla p\| \|\eta\| d\tau + \delta \varepsilon^2 \int_0^t \|\nabla^2 u\|^2 d\tau + \delta \int_0^t \|\nabla \Delta d\|^2 d\tau \\ &\quad + C(1 + Q(t)) \int_0^t (\|u\|_{\mathcal{H}^1}^2 + \|\nabla u\|^2) d\tau. \end{aligned} \quad (3.121)$$

Therefore, we complete the proof of lemma 3.6.  $\square$

Next, we can establish the following conormal estimates for the equivalent quantity  $\eta$ .

**Lemma 3.7.** *For  $m \geq 1$  and a smooth solution to (1.1) and (1.3), it holds that for  $\varepsilon \in (0, 1]$*

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} \|\eta(\tau)\|_{\mathcal{H}^{m-1}}^2 + \varepsilon \int_0^t \|\nabla \eta\|_{\mathcal{H}^{m-1}}^2 d\tau \\ & \leq C_{m+2} \left\{ (\|u_0\|_{\mathcal{H}^m}^2 + \|\nabla u_0\|_{\mathcal{H}^{m-1}}^2) + \delta \varepsilon^2 \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau + \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \right\} \\ & \quad + C_{m+2} \left\{ \int_0^t \|\nabla p\|_{\mathcal{H}^{m-1}} \|\eta\|_{\mathcal{H}^{m-1}} d\tau + C_\delta (1 + P(Q(t))) \int_0^t N_m(t) d\tau \right\}. \end{aligned} \quad (3.122)$$

*Proof.* The case for  $m = 1$  is already proved in Lemma 3.6. Assume that (3.122) is proved for  $k = m - 2$ . We shall prove that it holds for  $k = m - 1 \geq 1$ . For  $|\alpha| = m - 1$ , applying the operator  $\mathcal{Z}^\alpha$  to the equation (3.116), we find

$$\partial_t \mathcal{Z}^\alpha \eta + u \cdot \nabla \mathcal{Z}^\alpha \eta - \varepsilon \mathcal{Z}^\alpha \Delta \eta = \mathcal{Z}^\alpha F + \varepsilon \mathcal{Z}^\alpha (\Delta(\Pi B) \cdot u) + \mathcal{C}_2^\alpha, \quad (3.123)$$

where

$$\begin{aligned} \mathcal{C}_2^\alpha &= -[\mathcal{Z}^\alpha, u \cdot \nabla] \eta, \\ F &= \chi[F_1 \times n + \Pi(BF_2)] + \chi F_3 + F_4. \end{aligned}$$

Multiplying (3.123) by  $\mathcal{Z}^\alpha \eta$ , it is easy to deduce that

$$\begin{aligned} & \frac{1}{2} \int |\mathcal{Z}^\alpha \eta(t)|^2 dx - \frac{1}{2} \int |\mathcal{Z}^\alpha \eta_0|^2 dx \\ &= \varepsilon \int_0^t \int \mathcal{Z}^\alpha \Delta \eta \cdot \mathcal{Z}^\alpha \eta \, dx d\tau + \int_0^t \int \mathcal{Z}^\alpha F \cdot \mathcal{Z}^\alpha \eta \, dx d\tau \\ & \quad + \varepsilon \int_0^t \int \mathcal{Z}^\alpha (\Delta(\Pi B) \cdot u) \cdot \mathcal{Z}^\alpha \eta \, dx d\tau + \int_0^t \int \mathcal{C}_2^\alpha \cdot \mathcal{Z}^\alpha \eta \, dx d\tau. \end{aligned} \quad (3.124)$$

In the local basis, it holds that

$$\partial_j = \beta_j^1 \partial_{y_1} + \beta_j^2 \partial_{y_2} + \beta_j^3 \partial_z, \quad j = 1, 2, 3,$$

for harmless functions  $\beta_j^i$ ,  $i, j = 1, 2, 3$  depending on the boundary regularity and weight function  $\phi(z)$ . Therefore, the following commutation expansion holds:

$$\mathcal{Z}^\alpha \Delta \eta = \Delta \mathcal{Z}^\alpha \eta + \sum_{|\beta| \leq m-2} C_{1\beta} \partial_{zz} \mathcal{Z}^\beta \eta + \sum_{|\beta| \leq m-1} (C_{2\beta} \partial_z \mathcal{Z}^\beta \eta + C_{3\beta} Z_y \mathcal{Z}^\beta \eta).$$

Then, integrating by part and applying the Cauchy inequality, we obtain

$$\begin{aligned} & \varepsilon \int_0^t \int \mathcal{Z}^\alpha \Delta \eta \cdot \mathcal{Z}^\alpha \eta \, dx d\tau \\ &= \varepsilon \int_0^t \int \Delta \mathcal{Z}^\alpha \eta \cdot \mathcal{Z}^\alpha \eta \, dx d\tau + \sum_{|\beta| \leq m-2} \varepsilon \int_0^t \int C_{1\beta} \partial_{zz} \mathcal{Z}^\beta \eta \cdot \mathcal{Z}^\alpha \eta \, dx d\tau \\ & \quad + \sum_{|\beta| \leq m-1} \varepsilon \int_0^t \int (C_{2\beta} \partial_z \mathcal{Z}^\beta \eta + C_{3\beta} Z_y \mathcal{Z}^\beta \eta) \cdot \mathcal{Z}^\alpha \eta \, dx d\tau \\ & \leq -\frac{3}{4} \varepsilon \int_0^t \|\nabla \mathcal{Z}^\alpha \eta\|^2 d\tau + C \varepsilon \int_0^t \|\nabla \eta\|_{\mathcal{H}^{m-2}}^2 d\tau + C_{m+2} \varepsilon \int_0^t \|\eta\|_{\mathcal{H}^{m-1}}^2 d\tau. \end{aligned} \quad (3.125)$$

Note that there is no boundary term in the integrating by parts since  $\mathcal{Z}^\alpha \eta$  vanishes on the boundary. Substituting (3.125) into (3.124), we find

$$\begin{aligned} & \frac{1}{2} \int |\mathcal{Z}^\alpha \eta(t)|^2 dx + \frac{3}{4} \varepsilon \int_0^t \|\nabla \mathcal{Z}^\alpha \eta\|^2 d\tau \\ & \leq \frac{1}{2} \int |\mathcal{Z}^\alpha \eta_0|^2 dx + C\varepsilon \int_0^t \|\nabla \eta\|_{\mathcal{H}^{m-2}}^2 d\tau + C_{m+2}\varepsilon \int_0^t \|\eta\|_{\mathcal{H}^{m-1}}^2 d\tau \\ & \quad + \int_0^t \int \mathcal{Z}^\alpha F \cdot \mathcal{Z}^\alpha \eta dx d\tau + \varepsilon \int_0^t \int \mathcal{Z}^\alpha (\Delta(\Pi B) \cdot u) \cdot \mathcal{Z}^\alpha \eta dx d\tau \\ & \quad + \int_0^t \int \mathcal{C}_2^\alpha \cdot \mathcal{Z}^\alpha \eta dx d\tau. \end{aligned} \quad (3.126)$$

Similar to (3.118)-(3.119), we apply the Proposition 2.2 to deduce that

$$\begin{aligned} & \int_0^t \int \mathcal{Z}^\alpha (\chi F_1 \times n) \cdot \mathcal{Z}^\alpha \eta dx d\tau \leq C_m \left\{ \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta (1 + Q(t)) \int_0^t N_m(\tau) d\tau \right\}, \\ & \int_0^t \int \mathcal{Z}^\alpha (\chi \Pi(B F_2)) \cdot \mathcal{Z}^\alpha \eta dx d\tau \\ & \leq C_{m+1} \left\{ \int_0^t \|\nabla p\|_{\mathcal{H}^{m-1}} \|\eta\|_{\mathcal{H}^{m-1}} d\tau + (1 + Q(t)) \int_0^t N_m(\tau) d\tau \right\}, \\ & \int_0^t \int \mathcal{Z}^\alpha (\chi F_3) \cdot \mathcal{Z}^\alpha \eta dx d\tau \leq C_{m+2} \left\{ \delta \varepsilon^2 \int \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta (1 + Q(t)) \int_0^t N_m(\tau) d\tau \right\}, \end{aligned} \quad (3.127)$$

and

$$\int_0^t \int \mathcal{Z}^\alpha F_4 \cdot \mathcal{Z}^\alpha \eta dx d\tau \leq C_{m+1} \left\{ \delta \varepsilon^2 \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta (1 + Q(t)) \int_0^t N_m(\tau) d\tau \right\}. \quad (3.128)$$

Then, the combination of (3.127)-(3.128) gives directly

$$\begin{aligned} & \int_0^t \int \mathcal{Z}^\alpha F \cdot \mathcal{Z}^\alpha \eta dx d\tau \\ & \leq C_{m+2} \left\{ \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + \delta \varepsilon^2 \int \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau \right\} \\ & \quad + C_{m+2} \left\{ \int_0^t \|\nabla p\|_{\mathcal{H}^{m-1}} \|\eta\|_{\mathcal{H}^{m-1}} d\tau + C_\delta (1 + Q(t)) \int_0^t N_m(\tau) d\tau \right\}. \end{aligned} \quad (3.129)$$

Integrating by parts, one arrives at directly

$$\varepsilon \int_0^t \int \mathcal{Z}^\alpha (\Delta(\Pi B) \cdot u) \cdot \mathcal{Z}^\alpha \eta dx d\tau \leq \delta \varepsilon^2 \int_0^t \|\nabla \eta\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta C_{m+2} \int_0^t N_m(\tau) d\tau. \quad (3.130)$$

The remaining term are more involved, since it is desired to obtain an estimate independent of  $\partial_z \eta$ . First, it is easy to deduce that

$$\begin{aligned} \mathcal{C}_2^\alpha = & - \sum_{|\beta| \geq 1, \beta + \gamma = \alpha} \sum_{i=1}^2 C_{\alpha, \beta} \mathcal{Z}^\beta u_i \mathcal{Z}^\gamma \partial_{y_i} \eta - \sum_{|\beta| \geq 1, \beta + \gamma = \alpha} C_{\alpha, \beta} \mathcal{Z}^\beta (u \cdot N) \mathcal{Z}^\gamma \partial_z \eta \\ & - \sum_{|\beta| \leq m-2} C(\alpha, \beta, z) (u \cdot N) \partial_z \mathcal{Z}^\beta \eta, \end{aligned} \quad (3.131)$$

where  $C(\alpha, \beta, z)$  are smooth functions depending on  $\alpha, \beta$  and  $\varphi(z)$ . By virtue of the Proposition 2.2, we obtain

$$\begin{aligned} & \sum_{|\beta| \geq 1, \beta + \gamma = \alpha} \sum_{i=1}^2 C_{\alpha, \beta} \int_0^t \|\mathcal{Z}^\beta u_i \mathcal{Z}^\gamma \partial_{y_i} \eta\|^2 d\tau \\ & \leq C \|\mathcal{Z}u\|_{L_{x,t}^\infty}^2 \int_0^t \|Z_y \eta\|_{\mathcal{H}^{m-2}}^2 d\tau + C \|Z_y \eta\|_{L_{x,t}^\infty}^2 \int_0^t \|\mathcal{Z}u\|_{\mathcal{H}^{m-2}}^2 d\tau \end{aligned} \quad (3.132)$$

and

$$\begin{aligned} & \sum_{|\beta| \leq m-2} C(\alpha, \beta, z) \int_0^t \|(u \cdot N) \partial_z \mathcal{Z}^\beta \eta\|^2 d\tau \\ & \leq C \sum_{|\beta| \leq m-2} \left\| \frac{u \cdot N}{\varphi(z)} \right\|_{L_{x,t}^\infty}^2 \int_0^t \|Z_3 \mathcal{Z}^\beta \eta\|^2 d\tau \\ & \leq C \|u\|_{W_{x,t}^{1,\infty}}^2 \int_0^t \|\eta\|_{\mathcal{H}^{m-1}}^2 d\tau, \end{aligned} \quad (3.133)$$

where we have used the Hardy inequality  $\| \frac{u \cdot N}{\varphi} \|_{L^\infty} \leq C \|u\|_{W^{1,\infty}}$  due to the boundary condition  $u \cdot n|_{\partial\Omega} = 0$ . On the other hand, it is easy to deduce that

$$\begin{aligned} & - \sum_{|\beta| \geq 1, \beta + \gamma = \alpha} C_{\alpha, \beta} \mathcal{Z}^\beta (u \cdot N) \mathcal{Z}^\gamma \partial_z \eta \\ & = - \sum_{|\beta| \geq 1, \beta + \gamma = \alpha} C_{\alpha, \beta} \frac{1}{\varphi(z)} \mathcal{Z}^\beta (u \cdot N) \varphi(z) \mathcal{Z}^\gamma \partial_z \eta \\ & = \sum_{\tilde{\beta} \leq \beta, \tilde{\gamma} \leq \gamma} \mathcal{Z}^{\tilde{\beta}} \left( \frac{u \cdot N}{\varphi(z)} \right) \mathcal{Z}^{\tilde{\gamma}} (Z_3 \eta), \end{aligned}$$

where  $|\tilde{\beta}| + |\tilde{\gamma}| \leq m-1$ ,  $|\tilde{\gamma}| \leq m-2$ , and  $C_{\alpha, \tilde{\beta}, \tilde{\gamma}}$  are some smooth bounded functions depending on  $\varphi(z)$ . If  $\tilde{\beta} = 0$ , and hence  $|\tilde{\gamma}| \leq m-2$ , it holds that

$$\int_0^t \|\mathcal{Z}^{\tilde{\beta}} \left( \frac{u \cdot N}{\varphi(z)} \right) \mathcal{Z}^{\tilde{\gamma}} (Z_3 \eta)\|^2 d\tau \leq C \|u\|_{W_{x,t}^{1,\infty}}^2 \int_0^t \|\eta\|_{\mathcal{H}^{m-1}}^2 d\tau. \quad (3.134)$$

If  $\tilde{\beta} \neq 0$ , one attains that

$$\begin{aligned} & \int_0^t \|\mathcal{Z}^{\tilde{\beta}} \left( \frac{u \cdot N}{\varphi(z)} \right) \mathcal{Z}^{\tilde{\gamma}} (Z_3 \eta)\|^2 d\tau \\ & \leq \left\| \mathcal{Z} \left( \frac{u \cdot N}{\varphi(z)} \right) \right\|_{L_{x,t}^\infty}^2 \int_0^t \|Z_3 \eta\|_{\mathcal{H}^{m-2}}^2 d\tau + \|Z_3 \eta\|_{L_{x,t}^\infty}^2 \int_0^t \left\| \mathcal{Z} \left( \frac{u \cdot N}{\varphi(z)} \right) \right\|_{\mathcal{H}^{m-2}}^2 d\tau \\ & \leq \|\partial_z (u \cdot N)\|_{\mathcal{H}^{1,\infty}}^2 \int_0^t \|\eta\|_{\mathcal{H}^{m-1}}^2 d\tau + \|Z_3 \eta\|_{L_{x,t}^\infty}^2 \int_0^t \|\partial_z (u \cdot N)\|_{\mathcal{H}^{m-1}}^2 d\tau \\ & \leq C (\|Z_y u\|_{\mathcal{H}^{1,\infty}}^2 + \|Z_3 \eta\|_{L_{x,t}^\infty}^2) \int_0^t (\|\eta\|_{\mathcal{H}^{m-1}}^2 + \|u\|_{\mathcal{H}^m}^2) d\tau, \end{aligned} \quad (3.135)$$

where we have used the divergence free condition for the velocity and the Hardy inequality:

$$\sum_{|\beta| \leq 1} \|\mathcal{Z}^\beta \left( \frac{u \cdot N}{\varphi(z)} \right)\|_{\mathcal{H}^{m-2}} \leq C \|\partial_z (u \cdot N)\|_{\mathcal{H}^{m-1}}^2.$$

The combination of (3.131)-(3.135) yields directly

$$\int_0^t \int \mathcal{C}_2^\alpha \cdot \mathcal{Z}^\alpha \eta dx d\tau \leq C_m(1 + P(Q(t))) \int_0^t N_m(t) d\tau. \quad (3.136)$$

Substituting (3.129), (3.130) and (3.136) into (3.126), we find

$$\begin{aligned} & \frac{1}{2} \int |\mathcal{Z}^\alpha \eta(t)|^2 dx + \frac{3\varepsilon}{4} \int_0^t \|\nabla \mathcal{Z}^\alpha \eta\|^2 d\tau \\ & \leq C_{m+2} \left\{ \frac{1}{2} \int |\mathcal{Z}^\alpha \eta_0|^2 dx + C\varepsilon \int_0^t \|\nabla \eta\|_{\mathcal{H}^{m-2}}^2 d\tau + \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \right\} \\ & \quad + C_{m+2} \left\{ \delta \varepsilon^2 \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau + \int_0^t \|\nabla p\|_{\mathcal{H}^{m-1}} \|\eta\|_{\mathcal{H}^{m-1}} d\tau \right\} \\ & \quad + C_{m+2} C_\delta (1 + P(Q(t))) \int_0^t N_m(t) d\tau. \end{aligned}$$

By the induction assumption, one can eliminate the term  $\varepsilon \int_0^t \|\nabla \eta\|_{\mathcal{H}^{m-2}}^2 d\tau$ . Therefore, we complete the proof of the Lemma 3.7.  $\square$

Similar to the analysis of (3.106)-(3.107), it is easy to deduce that

$$\int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau \leq C_{m+2} \left\{ \int_0^t \|\nabla \eta\|_{\mathcal{H}^{m-1}}^2 d\tau + \int_0^t \|\nabla u\|_{\mathcal{H}^m}^2 d\tau + \int_0^t N_m(\tau) d\tau \right\},$$

which, together with (3.15), (3.90) and (3.122), yields directly

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} (\|(u, \nabla d)\|_{\mathcal{H}^m}^2 + \|(\nabla u, \Delta d)\|_{\mathcal{H}^{m-1}}^2) + \varepsilon \int_0^t \|\nabla u\|_{\mathcal{H}^m}^2 d\tau \\ & \quad + \varepsilon \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau + \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau + \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \\ & \leq C_{m+2} \left\{ \|(u_0, \nabla d_0)\|_{\mathcal{H}^m}^2 + \|(\nabla u_0, \Delta d_0)\|_{\mathcal{H}^{m-1}}^2 + \int_0^t \|\nabla p\|_{\mathcal{H}^{m-1}} \|\eta\|_{\mathcal{H}^{m-1}} d\tau \right. \\ & \quad \left. + \int_0^t \|\nabla^2 p_1\|_{\mathcal{H}^{m-1}} \|u\|_{\mathcal{H}^m} d\tau + \varepsilon^{-1} \int_0^t \|\nabla p_2\|_{\mathcal{H}^{m-1}}^2 d\tau + (1 + P(Q(t))) \int_0^t N_m(t) d\tau \right\}. \end{aligned} \quad (3.137)$$

#### 3.4. Estimates for Pressure

It remains to estimate the pressure and the  $L^\infty$ -norm on the right-hand side of the estimates of Lemmas 3.3, 3.5 and 3.7. The aim of this section is to give the estimate for the pressure.

**Lemma 3.8.** *For  $m \geq 2$ , we have the following estimates for the pressure:*

$$\begin{aligned} & \int_0^t \|\nabla p_1\|_{\mathcal{H}^{m-1}}^2 d\tau + \int_0^t \|\nabla^2 p_1\|_{\mathcal{H}^{m-1}}^2 d\tau \\ & \leq C_{m+2} Q(t) \int_0^t N_m(\tau) d\tau + C_{m+2} Q(t) \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau, \end{aligned} \quad (3.138)$$

and

$$\int_0^t \|\nabla p_2\|_{\mathcal{H}^{m-1}}^2 d\tau \leq C_{m+2} \varepsilon \int_0^t N_m(\tau) d\tau. \quad (3.139)$$

*Proof.* We recall that we have  $p = p_1 + p_2$ , where

$$\Delta p_1 = -\operatorname{div}(u \cdot \nabla u + \nabla d \cdot \Delta d), \quad x \in \Omega, \quad \partial_n p_1 = -(u \cdot \nabla u) \cdot n, \quad x \in \partial\Omega, \quad (3.140)$$

and

$$\Delta p_2 = 0, \quad x \in \Omega, \quad \partial_n p_2 = \varepsilon \Delta u \cdot n, \quad x \in \partial\Omega. \quad (3.141)$$

For  $|\alpha_0| + |\alpha_1| = m - 1$ , by virtue of the standard elliptic regularity results with Neumann boundary conditions, we get that

$$\begin{aligned} & \|\nabla \partial_t^{\alpha_0} p_1\|_{|\alpha_1|} + \|\nabla^2 \partial_t^{\alpha_0} p_1\|_{|\alpha_1|} \\ & \leq C(\|\partial_t^{\alpha_0} \operatorname{div}(u \cdot \nabla u + \nabla d \cdot \Delta d)\|_{|\alpha_1|} + \|\partial_t^{\alpha_0}(u \cdot \nabla u + \nabla d \cdot \Delta d)\|) \\ & \quad + C|\partial_t^{\alpha_0}(u \cdot \nabla u) \cdot n|_{H^{|\alpha_1|+\frac{1}{2}}(\partial\Omega)}. \end{aligned} \quad (3.142)$$

Since  $u \cdot n = 0$  on the boundary, we note that

$$(u \cdot \nabla u) \cdot n = -(u \cdot \nabla n) \cdot u, \quad x \in \partial\Omega,$$

and hence that

$$|\partial_t^{\alpha_0}(u \cdot \nabla u) \cdot n|_{H^{|\alpha_1|+\frac{1}{2}}(\partial\Omega)} \leq C_{m+2} |\partial_t^{\alpha_0}(u \otimes u)|_{H^{|\alpha_1|+\frac{1}{2}}(\partial\Omega)}. \quad (3.143)$$

In view of the trace theorem in Proposition 2.3, one arrives at

$$|\partial_t^{\alpha_0}(u \otimes u)|_{H^{|\alpha_1|+\frac{1}{2}}(\partial\Omega)} \lesssim \|\nabla \partial_t^{\alpha_0}(u \otimes u)\|_{|\alpha_1|} + \|\partial_t^{\alpha_0}(u \otimes u)\|_{|\alpha_1|+1},$$

which, together with (3.143), reads

$$|\partial_t^{\alpha_0}(u \cdot \nabla u) \cdot n|_{H^{|\alpha_1|+\frac{1}{2}}(\partial\Omega)} \leq C_{m+2} (\|u \cdot \nabla u\|_{\mathcal{H}^{m-1}} + \|u \otimes u\|_{\mathcal{H}^m}). \quad (3.144)$$

Substituting (3.144) into (3.142) and integrating the resulting inequality over  $[0, t]$ , we find

$$\begin{aligned} & \int_0^t \|\nabla p_1\|_{\mathcal{H}^{m-1}}^2 d\tau + \int_0^t \|\nabla^2 p_1\|_{\mathcal{H}^{m-1}}^2 d\tau \\ & = \int_0^t \|\nabla \partial_t^{\alpha_0} p_1\|_{|\alpha_1|}^2 d\tau + \int_0^t \|\nabla^2 \partial_t^{\alpha_0} p_1\|_{|\alpha_1|}^2 d\tau \\ & \leq C_{m+2} \left\{ \int_0^t \|\nabla u \cdot \nabla u\|_{\mathcal{H}^{m-1}}^2 d\tau + \int_0^t \|u \cdot \nabla u\|_{\mathcal{H}^{m-1}}^2 d\tau + \int_0^t \|u \otimes u\|_{\mathcal{H}^m}^2 d\tau \right\} \\ & \quad + C_{m+2} \left\{ \int_0^t \|\Delta d \cdot \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + \int_0^t \|\nabla d \cdot \nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \right\} \\ & \leq C_{m+2} (\|u\|_{W_{x,t}^{1,\infty}}^2 + \|\Delta d\|_{L_{x,t}^\infty}^2) \int_0^t (\|u\|_{\mathcal{H}^m}^2 + \|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|\Delta d\|_{\mathcal{H}^{m-1}}^2) d\tau \\ & \quad + C_{m+2} \int_0^t \|\nabla d \cdot \nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau, \end{aligned} \quad (3.145)$$

where we have used the Proposition 2.2 in the last inequality. On the other hand, the application of Proposition 2.2 yields

$$\begin{aligned}
& \int_0^t \|\nabla d \cdot \nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \\
&= \sum_{|\beta| \geq 1, |\beta|+|\gamma|=m-1} \int_0^t \|\mathcal{Z}^{\beta-1} \mathcal{Z} \nabla d \cdot \mathcal{Z}^\gamma \nabla \Delta d\|^2 d\tau + \int_0^t \|\nabla d \cdot \mathcal{Z}^\alpha \nabla \Delta d\|^2 d\tau \\
&\leq \|\mathcal{Z} \nabla d\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-2}}^2 d\tau + \|\nabla \Delta d\|_{L_{x,t}^\infty}^2 \int_0^t \|\mathcal{Z} \nabla d\|_{\mathcal{H}^{m-2}}^2 d\tau \\
&\quad + \|\nabla d\|_{L_{x,t}^\infty}^2 \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \\
&\leq C_1 Q(t) \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + Q(t) \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau,
\end{aligned}$$

which, together with (3.145), reads

$$\int_0^t \|\nabla p_1\|_{\mathcal{H}^{m-1}}^2 d\tau + \int_0^t \|\nabla^2 p_1\|_{\mathcal{H}^{m-1}}^2 d\tau \leq C_{m+2} Q(t) \int_0^t N_m(t) d\tau + C_{m+2} Q(t) \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau.$$

Hence, we complete the proof of estimate (3.138). It remains to estimate  $p_2$ . By using the elliptic regularity for the Neumann problem again, we get that for  $|\alpha_0| + |\alpha_1| = m - 1$ ,

$$\|\nabla \partial_t^{\alpha_0} p_2\|_{|\alpha_1|} \leq C \varepsilon |\partial_t^{\alpha_0} \Delta u \cdot n|_{H^{|\alpha_1|-\frac{1}{2}}(\partial\Omega)}. \quad (3.146)$$

From the results by Masmoudi and Rousset [14], we have the following fact that

$$|\partial_t^{\alpha_0} \Delta u \cdot n|_{H^{|\alpha_1|-\frac{1}{2}}(\partial\Omega)} \leq C_{m+2} |\partial_t^{\alpha_0} u|_{H^{|\alpha_1|+\frac{1}{2}}(\partial\Omega)}. \quad (3.147)$$

Applying the trace inequality in Proposition 2.3, we get

$$|\partial_t^{\alpha_0} u|_{H^{|\alpha_1|+\frac{1}{2}}(\partial\Omega)} \leq C (\|\nabla \partial_t^{\alpha_0} u\|_{|\alpha_1|} + \|\partial_t^{\alpha_0} u\|_{|\alpha_1|+1}). \quad (3.148)$$

Substituting (3.147)- (3.148) into (3.146) and integrating the resulting inequality over  $[0, t]$ , one arrives at

$$\int_0^t \|\nabla p_2\|_{\mathcal{H}^{m-1}}^2 d\tau = \int_0^t \|\nabla \partial_t^{\alpha_0} p_2\|_{|\alpha_1|}^2 d\tau \leq C_{m+2} \varepsilon \int_0^t (\|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|u\|_{\mathcal{H}^m}^2) d\tau.$$

Hence, we complete the proof of the estimate (3.139).  $\square$

By virtue of (3.138), the Hölder and Cauchy inequalities, it is easy to deduce that

$$\begin{aligned}
& \int_0^t \|\nabla^2 p_1\|_{\mathcal{H}^{m-1}} \|u\|_{\mathcal{H}^m} d\tau \\
&\leq \left( \int_0^t \|\nabla^2 p_1\|_{\mathcal{H}^{m-1}}^2 d\tau \right)^{\frac{1}{2}} \left( \int_0^t \|u\|_{\mathcal{H}^m}^2 d\tau \right)^{\frac{1}{2}} \\
&\leq C_{m+2} Q(t)^{\frac{1}{2}} \left( \int_0^t N_m(\tau) d\tau \right)^{\frac{1}{2}} \left( \int_0^t \|u\|_{\mathcal{H}^m}^2 d\tau \right)^{\frac{1}{2}} \\
&\quad + C_{m+2} Q(t)^{\frac{1}{2}} \left( \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \right)^{\frac{1}{2}} \left( \int_0^t \|u\|_{\mathcal{H}^m}^2 d\tau \right)^{\frac{1}{2}} \\
&\leq \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_{m+2} C_\delta Q(t) \int_0^t N_m(\tau) d\tau.
\end{aligned} \quad (3.149)$$



Similarly, we also deduce

$$\int_0^t \|\nabla p\|_{\mathcal{H}^{m-1}} \|\eta\|_{\mathcal{H}^{m-1}} d\tau \leq \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_{m+2} C_\delta Q(t) \int_0^t N_m(\tau) d\tau. \quad (3.150)$$

Substituting (3.138), (3.139), (3.149) and (3.150) into (3.137) and choosing  $\delta$  small enough, one arrives at

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} (\|(u, \nabla d)\|_{\mathcal{H}^m}^2 + \|(\nabla u, \Delta d)\|_{\mathcal{H}^{m-1}}^2) + \varepsilon \int_0^t \|\nabla u\|_{\mathcal{H}^m}^2 d\tau \\ & + \varepsilon \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau + \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau + \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \\ & \leq C_{m+2} \left\{ \|(u_0, \nabla d_0)\|_{\mathcal{H}^m}^2 + \|(\nabla u_0, \Delta d_0)\|_{\mathcal{H}^{m-1}}^2 + C(1 + P(Q(t))) \int_0^t N_m(t) d\tau \right\}. \end{aligned} \quad (3.151)$$

### 3.5. $L^\infty$ – estimates

In this subsection, we shall provide the  $L^\infty$ –estimates of  $(u, d)$  which are needed to estimate on the right-hand side of the estimate (3.151).

**Lemma 3.9.** *For a smooth solution to (1.1) and (1.3), it holds that*

$$\|(u, \nabla d)\|_{L^\infty}^2 \leq C_3 N_m(t), \quad m \geq 2, \quad (3.152)$$

$$\|u\|_{\mathcal{H}^{2,\infty}}^2 \leq C N_m(t), \quad m \geq 4, \quad (3.153)$$

$$\|(u_t, d_t, \nabla d_t)\|_{L^\infty}^2 \leq C_3 N_m^3(t), \quad m \geq 3, \quad (3.154)$$

$$\|\nabla^2 d\|_{L^\infty}^2 \leq C_4 N_m^3(t), \quad m \geq 3, \quad (3.155)$$

$$\|\nabla \Delta d\|_{L^\infty}^2 \leq C_4 N_m^3(t), \quad m \geq 3. \quad (3.156)$$

*Proof.* By virtue of the Sobolev inequality in Proposition 2.3, one arrives at

$$\|u\|_{L^\infty}^2 \leq C(\|\nabla u\|_1^2 + \|u\|_2^2) \quad (3.157)$$

and

$$\|\nabla d\|_{L^\infty}^2 \leq C(\|\nabla^2 d\|_1^2 + \|\nabla d\|_2^2). \quad (3.158)$$

In view of the standard elliptic regularity results with Neumann boundary condition, we get that

$$\|\nabla^2 d\|_m^2 \leq C_{m+2}(\|\Delta d\|_m^2 + \|\nabla d\|^2). \quad (3.159)$$

Then, the combination of (3.158) and (3.159) yields directly

$$\|\nabla d\|_{L^\infty}^2 \leq C_3(\|\Delta d\|_1^2 + \|\nabla d\|_2^2),$$

which, together with (3.157), complete the proof of (3.152). The estimates (3.153) follows directly from the application of Sobolev inequality in Proposition 2.3. In view of Sobolev inequality in Proposition 2.3, one arrives at

$$\|u_t\|_{L^\infty}^2 \leq C(\|\nabla u_t\|_1^2 + \|u_t\|_2^2) \leq C N_m(t), \quad \text{for } m \geq 3. \quad (3.160)$$

By virtue of the equation (1.1)<sub>2</sub>, we find

$$\begin{aligned} \|d_t\|_{L^\infty}^2 &\leq C(\|\nabla d_t\|_1^2 + \|d_t\|_2^2) \\ &\leq C(\|\nabla d_t\|_1^2 + \|\Delta d\|_2^2 + \|u \cdot \nabla d\|_2^2 + \|\nabla d\|_2^2). \end{aligned} \quad (3.161)$$

By Proposition 2.2 and estimate (3.152), one attains

$$\|u \cdot \nabla d\|_2^2 \leq C(\|u\|_{L^\infty}^2 \|\nabla d\|_2^2 + \|\nabla d\|_{L^\infty}^2 \|u\|_2^2) \leq C_3 N_m(t), \text{ for } m \geq 2; \quad (3.162)$$

and

$$\begin{aligned} \|\nabla d\|_2^2 &\leq \sum_{|\gamma| \geq 1, |\beta| + |\gamma| \leq 2} \int |Z^\beta(|\nabla d|^2) Z^\gamma d|^2 dx + \|\nabla d\|_2^2 \\ &\leq \|Zd\|_{L^\infty}^2 \|\nabla d\|_1^2 + \|\nabla d\|_{L^\infty}^2 \|Zd\|_1^2 + \|\nabla d\|_{L^\infty}^2 \|\nabla d\|_2^2 \\ &\leq C_3 N_2^3(t). \end{aligned} \quad (3.163)$$

Hence, the combination of (3.161)-(3.163) gives directly

$$\|d_t\|_{L^\infty}^2 \leq C_3 N_m^3(t), \text{ for } m \geq 3. \quad (3.164)$$

The application of Proposition 2.3 and the standard elliptic regularity results with Neumann boundary condition, we obtain for  $m \geq 3$

$$\|\nabla d_t\|_{L^\infty}^2 \leq C(\|\nabla^2 d_t\|_1^2 + \|\nabla d_t\|_2^2) \leq C(\|\Delta d_t\|_1^2 + \|\nabla d_t\|_2^2) \leq C(\|\Delta d\|_{\mathcal{H}^2}^2 + \|\nabla d\|_{\mathcal{H}^3}^2). \quad (3.165)$$

Then, the combination of (3.160), (3.164) and (3.165) completes the proof of (3.154). It is easy to deduce that

$$\begin{aligned} \partial_{ii} &= \partial_{y_i}^2 - \partial_{y_i}(\partial_i \psi \partial_z) - \partial_i \psi \partial_z \partial_{y_i} + (\partial_i \psi)^2 \partial_z^2, \quad i = 1, 2, \\ \partial_1 \partial_2 &= \partial_{y_1} \partial_{y_2} - \partial_{y_2}(\partial_1 \psi \partial_z) - \partial_2 \psi \partial_{y_1} \partial_z + \partial_2 \psi \partial_1 \psi \partial_z^2, \\ \partial_i \partial_3 &= \partial_{y_i} \partial_z - \partial_i \psi \partial_z^2, \quad i = 1, 2. \end{aligned}$$

Then, we find that

$$\Delta = (1 + |\nabla \psi|^2) \partial_z^2 + \sum_{i=1,2} (\partial_{y_i}^2 - \partial_{y_i}(\partial_i \psi \partial_z) - \partial_i \psi \partial_z \partial_{y_i}). \quad (3.166)$$

and

$$\begin{aligned} \nabla^2 &= [(1 + |\nabla \psi|^2) + \partial_2 \psi \partial_1 \psi - \partial_1 \psi - \partial_2 \psi] \partial_z^2 + \partial_{y_1} \partial_{y_2} \\ &\quad + \sum_{i=1,2} (\partial_{y_i}^2 - \partial_{y_i}(\partial_i \psi \partial_z) - \partial_i \psi \partial_z \partial_{y_i}) - \partial_{y_2}(\partial_1 \psi \partial_z) \\ &\quad - \partial_2 \psi \partial_{y_1} \partial_z + \partial_{y_1} \partial_z + \partial_{y_2} \partial_z. \end{aligned} \quad (3.167)$$

The combination of (3.166) and (3.167) and Proposition 2.3 yield that

$$\begin{aligned} \|\nabla^2 d\|_{L^\infty}^2 &\leq C_1(\|\Delta d\|_{L^\infty}^2 + \|\partial_z \partial_{y_i} d\|_{L^\infty}^2 + \|\partial_{y_i} \partial_{y_j} d\|_{L^\infty}^2) \\ &\leq C_1(\|\nabla \Delta d\|_1^2 + \|\Delta d\|_2^2 + \|\nabla \partial_z \partial_{y_i} d\|_1^2 + \|\partial_z \partial_{y_i} d\|_2^2) \\ &\quad + C(\|\nabla \partial_{y_i} \partial_{y_j} d\|_1^2 + \|\partial_{y_i} \partial_{y_j} d\|_2^2) \\ &\leq C_1(\|\nabla \Delta d\|_1^2 + \|\Delta d\|_2^2 + \|\nabla^2 d\|_2^2 + \|\nabla d\|_3^2) \\ &\leq C_4(\|\nabla \Delta d\|_1^2 + \|\Delta d\|_2^2 + \|\nabla d\|_3^2), \end{aligned} \quad (3.168)$$

where we have used the estimate (3.159) in the last inequality. In order to deal with the first term on the right hand side of (3.168), we apply the equation (1.1)<sub>2</sub> to attain that

$$\begin{aligned}\|\nabla \Delta d\|_1^2 &\leq \|\nabla(d_t + u \cdot \nabla d - |\nabla d|^2 d)\|_1^2 \\ &\leq \|\nabla d_t\|_1^2 + \|\nabla u \cdot \nabla d\|_1^2 + \|u \cdot \nabla^2 d\|_1^2 + \|\nabla(|\nabla d|^2 d)\|_1^2.\end{aligned}\quad (3.169)$$

It is easy to deduce that

$$\|\nabla d_t\|_1^2 \leq \|\nabla d\|_{\mathcal{H}^2}^2 \leq N_m(t), \text{ for } m \geq 2, \quad (3.170)$$

$$\|\nabla u \cdot \nabla d\|_1^2 \leq \|(\nabla u, \nabla d)\|_{L^\infty}^2 \|(\nabla u, \nabla d)\|_1^2 \leq C_3 N_m^2(t), \text{ for } m \geq 2, \quad (3.171)$$

and

$$\|u \cdot \nabla^2 d\|_1^2 \leq \|u\|_{L^\infty}^2 \|\nabla^2 d\|^2 + \|Zu\|_{L^\infty}^2 \|\nabla^2 d\|^2 + \|u\|_{L^\infty}^2 \|\nabla^2 d\|_1^2 \leq C N_m^2(t), \text{ for } m \geq 3. \quad (3.172)$$

In view of the basic fact  $|d| = 1$  (see (3.7)), one arrives at

$$\|\nabla(|\nabla d|^2 d)\|_1^2 \leq C N_m^3(t), \text{ for } m \geq 3. \quad (3.173)$$

Substituting (3.170)-(3.173) into (3.169), we find

$$\|\nabla \Delta d\|_1^2 \leq C_3 N_m^3(t), \quad m \geq 3,$$

which, together with (3.168), completes the proof of (3.155). By virtue of the (1.1)<sub>2</sub>, (3.152) and (3.155), one attains for  $m \geq 3$

$$\begin{aligned}\|\nabla \Delta d\|_{L^\infty}^2 &\leq \|\nabla(d_t + u \cdot \nabla d - |\nabla d|^2 d)\|_{L^\infty}^2 \\ &\leq \|\nabla d_t\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty}^2 \|\nabla d\|_{L^\infty}^2 + \|u\|_{L^\infty}^2 \|\nabla^2 d\|_{L^\infty}^2 \\ &\quad + \|\nabla d\|_{L^\infty}^2 \|\nabla^2 d\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^4 \|\nabla d\|_{L^\infty}^2 \\ &\leq \|\nabla d_t\|_{L^\infty}^2 + C_4 N_m^3(t),\end{aligned}$$

which, together with (3.165), completes the proof of estimate (3.156).  $\square$

In order to give the estimate for  $\|\nabla u\|_{\mathcal{H}^{1,\infty}}$ , we need the lemma as follows, refer to [14] or [16].

**Lemma 3.10.** *Consider  $\rho$  a smooth solution of*

$$\partial_t \rho + u \cdot \nabla \rho = \varepsilon \partial_{zz} \rho + \mathcal{S}, \quad z > 0, \quad \rho(t, y, 0) = 0 \quad (3.174)$$

for some smooth divergence free vector field  $u$  such that  $u \cdot n$  vanishes on the boundary. Assume that  $\rho$  and  $\mathcal{S}$  are compact supported in  $z$ . Then, we have the estimate:

$$\|\rho(t)\|_{\mathcal{H}^{1,\infty}} \leq C \|\rho_0\|_{\mathcal{H}^{1,\infty}} + C \int_0^t ((\|u\|_{\mathcal{H}^{2,\infty}} + \|\partial_z u\|_{\mathcal{H}^{1,\infty}})(\|\rho\|_{\mathcal{H}^{1,\infty}} + \|\rho\|_{\mathcal{H}^{m_0+3}}) + \|\mathcal{S}\|_{\mathcal{H}^{1,\infty}}) d\tau \quad (3.175)$$

for  $m_0 \geq 2$ .

Finally, one gives the estimate for the quantity  $\|\nabla u\|_{\mathcal{H}^{1,\infty}}$ .

**Lemma 3.11.** *For  $m \geq 6$ , we have the estimate*

$$\|\nabla u\|_{\mathcal{H}^{1,\infty}}^2 \leq C_{m+2} \left\{ N_m(0) + \|u\|_{\mathcal{H}^m}^2 + \delta \varepsilon \int_0^t \|\nabla^2 u\|_{\mathcal{H}^4}^2 d\tau + C_\delta \int_0^t N_m^4(\tau) d\tau \right\}. \quad (3.176)$$

*Proof.* Away from the boundary, we clearly have by the classical isotropic Sobolev embedding that

$$\|\chi \nabla u\|_{\mathcal{H}^{1,\infty}} \lesssim \|u\|_{\mathcal{H}^m}, \quad m \geq 4, \quad (3.177)$$

where the support of  $\chi$  is away from the boundary. Consequently, by using a partition of unity subordinated to the covering we only have to estimate  $\|\chi_j \nabla u\|_{\mathcal{H}^{1,\infty}}$ ,  $j \geq 1$ . For notational convenience, we shall denote  $\chi_j$  by  $\chi$ . Similar to [14], we use the local parametrization in the neighborhood of the boundary given by a normal geodesic system in which the Laplacian takes a convenient form. Denote

$$\Psi^n(y, z) = \begin{pmatrix} y \\ \psi(y) \end{pmatrix} - zn(y) = x,$$

where

$$n(y) = \frac{1}{\sqrt{1 + |\nabla \psi(y)|^2}} \begin{pmatrix} \partial_1 \psi(y) \\ \partial_2 \psi(y) \\ -1 \end{pmatrix}$$

is the unit outward normal. As before, one can extend  $n$  and  $\Pi$  in the interior by setting

$$n(\Psi^n(y, z)) = n(y), \quad \Pi(\Psi^n(y, z)) = \Pi(y) = I - n \otimes n,$$

where  $I$  is the unit matrix. Note that  $n(y, z)$  and  $\Pi(y, z)$  have different definitions from the ones used before. The advantages of this parametrization is that in the associated local basis  $(e_{y_1}, e_{y_2}, e_z)$  of  $\mathbb{R}^3$ , it holds that  $\partial_z = \partial_n$  and

$$(e_{y_i})|_{\Psi^n(y, z)} \cdot (e_z)|_{\Psi^n(y, z)} = 0, \quad i = 1, 2.$$

The scalar product on  $\mathbb{R}^3$  induces in this coordinate system the Riemannian metric  $g$  with the norm

$$g(y, z) = \begin{pmatrix} \tilde{g}(y, z) & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, the Laplacian in this coordinate system has the form

$$\Delta f = \partial_{zz} f + \frac{1}{2} \partial_z (\ln |g|) \partial_z f + \Delta_{\tilde{g}} f, \quad (3.178)$$

where  $|g|$  denotes the determinant of the matrix  $g$ , and  $\Delta_{\tilde{g}}$  is defined by

$$\Delta_{\tilde{g}} f = \frac{1}{\sqrt{|\tilde{g}|}} \sum_{i,j=1,2} \partial_{y_i} (\tilde{g}^{ij} |\tilde{g}|^{\frac{1}{2}} \partial_{y_j} f),$$

which only involves the tangential derivatives and  $\{\tilde{g}^{ij}\}$  is the inverse matrix to  $g$ .

Next, thanks to (3.106) (in the coordinate system that we have just defined) and Lemma 3.9, we have for  $m \geq 4$

$$\begin{aligned} \|\chi \nabla u\|_{\mathcal{H}^{1,\infty}}^2 &\leq C_3 (\|\chi \Pi \partial_n u\|_{\mathcal{H}^{1,\infty}}^2 + \|u\|_{\mathcal{H}^{2,\infty}}^2) \\ &\leq C_3 (\|\chi \Pi \partial_n u\|_{\mathcal{H}^{1,\infty}}^2 + N_m(t)). \end{aligned} \quad (3.179)$$

Consequently, it suffices to estimate  $\|\chi \Pi \partial_n u\|_{\mathcal{H}^{1,\infty}}$ . To this end, it is useful to use the vorticity  $w = \nabla \times u$ , see [13, 14, 16]. Indeed, it is easy to deduce that

$$\Pi(w \times n) = \Pi((\nabla u - \nabla u^t) \cdot n) = \Pi(\partial_n u - \nabla(u \cdot n) + \nabla n^t \cdot u),$$

which implies

$$\begin{aligned}\|\chi\Pi\partial_n u\|_{\mathcal{H}^{1,\infty}}^2 &\leq C_3(\|\chi\Pi(w \times n)\|_{\mathcal{H}^{1,\infty}}^2 + \|u\|_{\mathcal{H}^{2,\infty}}^2) \\ &\leq C_3(\|\chi\Pi(w \times n)\|_{\mathcal{H}^{1,\infty}}^2 + N_m(t)),\end{aligned}\quad (3.180)$$

where we have used the Lemma 3.9 in the last inequality. In other words, we only need to estimate  $\|\chi\Pi(w \times n)\|_{\mathcal{H}^{1,\infty}}$ . It is easy to see that  $w$  solves the vorticity equation

$$w_t + (u \cdot \nabla)w - \varepsilon\Delta w = F_1, \quad (3.181)$$

where  $F_1 \triangleq w \cdot \nabla u - \nabla \times (\nabla d \cdot \Delta d)$ . In the support of  $\chi$ , let

$$\tilde{w}(y, z) = w(\Psi^n(y, z)), \quad \tilde{u}(y, z) = u(\Psi^n(y, z)), \quad \tilde{d}(y, z) = d(\Psi^n(y, z)),$$

The combination of (3.178) and (3.181) yields directly

$$\partial_t \tilde{w} + \tilde{u}^1 \partial_{y_1} \tilde{w} + \tilde{u}^2 \partial_{y_2} \tilde{w} + \tilde{u} \cdot n \partial_z \tilde{w} = \varepsilon(\partial_{zz} \tilde{w} + \frac{1}{2} \partial_z (\ln |g|) \partial_z \tilde{w} + \Delta_{\tilde{g}} \tilde{w}) + \tilde{F}_1 \quad (3.182)$$

and

$$\partial_t \tilde{u} + \tilde{u}^1 \partial_{y_1} \tilde{u} + \tilde{u}^2 \partial_{y_2} \tilde{u} + \tilde{u} \cdot n \partial_z \tilde{u} = \varepsilon(\partial_{zz} \tilde{u} + \frac{1}{2} \partial_z (\ln |g|) \partial_z \tilde{u} + \Delta_{\tilde{g}} \tilde{u}) + \tilde{F}_2, \quad (3.183)$$

where  $\tilde{F}_2 = F_2(\Psi^n(y, z))$  and  $F_2 \triangleq -\nabla p - \nabla d \cdot \Delta d$ . Similar to (3.110), we define

$$\tilde{\eta} = \chi(\tilde{w} \times n + \Pi(B\tilde{u})).$$

It is easy to deduce that  $\tilde{\eta}$  satisfies

$$\tilde{\eta}(y, 0) = 0.$$

and solves the equation

$$\begin{aligned}\partial_t \tilde{\eta} + \tilde{u}^1 \partial_{y_1} \tilde{\eta} + \tilde{u}^2 \partial_{y_2} \tilde{\eta} + \tilde{u} \cdot n \partial_z \tilde{\eta} \\ = \varepsilon \left( \partial_{zz} \tilde{\eta} + \frac{1}{2} \partial_z (\ln |g|) \partial_z \tilde{\eta} \right) + \chi(\tilde{F}_1 \times n) + \chi\Pi(B\tilde{F}_2) + F^\chi + \chi F^\kappa,\end{aligned}\quad (3.184)$$

where the source terms are given by

$$\begin{aligned}F^\chi &= [(\tilde{u}^1 \partial_{y_1} + \tilde{u}^2 \partial_{y_2} + \tilde{u} \cdot n \partial_z) \chi] (\tilde{w} \times n + \Pi(Bu)) \\ &\quad - \varepsilon \left( \partial_{zz} \chi + 2 \partial_z \chi \partial_z + \frac{1}{2} \partial_z (\ln |g|) \partial_z \chi \right) (\tilde{w} \times n + \Pi(Bu)),\end{aligned}\quad (3.185)$$

and

$$\begin{aligned}F^\kappa &= (\tilde{u}^1 \partial_{y_1} \Pi + \tilde{u}^2 \partial_{y_2} \Pi) \cdot (B\tilde{u}) + w \times (\tilde{u}^1 \partial_{y_1} n + \tilde{u}^2 \partial_{y_2} n) \\ &\quad + \Pi [(\tilde{u}^1 \partial_{y_1} + \tilde{u}^2 \partial_{y_2} + u \cdot n \partial_z) B \cdot u] + \varepsilon \Delta_{\tilde{g}} \tilde{w} \times n + \varepsilon \Pi(B \Delta_{\tilde{g}} \tilde{u}).\end{aligned}\quad (3.186)$$

Note that in the derivation of the source terms above, in particular,  $F^\kappa$ , which contains all the commutators coming from the fact that  $n$  and  $\Pi$  are not constant, we have used the fact that in the coordinate system just defined,  $n$  and  $\Pi$  do not depend on the normal variable. Since  $\Delta_{\tilde{g}}$  involves only the tangential derivatives, and the derivatives of  $\chi$  are compactly supported away from the boundary, the following estimates hold for  $m \geq 6$

$$\|F^\chi\|_{\mathcal{H}^{1,\infty}}^2 \leq C_3(\|u\|_{\mathcal{H}^{1,\infty}}^2 \|u\|_{\mathcal{H}^{2,\infty}}^2 + \varepsilon \|u\|_{\mathcal{H}^{3,\infty}}^2) \leq C_3 N_m^2(t), \quad (3.187)$$

$$\|\chi(\tilde{F}^1 \times n)\|_{\mathcal{H}^{1,\infty}}^2 \leq C_2(\|\nabla u\|_{\mathcal{H}^{1,\infty}}^2 + \|\nabla d\|_{\mathcal{H}^{1,\infty}}^2 \|\nabla \Delta d\|_{\mathcal{H}^{1,\infty}}^2) \leq C_2 N_m^4(t), \quad (3.188)$$

$$\|\chi \Pi(B\tilde{F}_2)\|_{\mathcal{H}^{1,\infty}}^2 \leq C_3(\|\nabla p\|_{\mathcal{H}^{1,\infty}}^2 + \|\nabla d\|_{\mathcal{H}^{1,\infty}}^2 \|\Delta d\|_{\mathcal{H}^{1,\infty}}^2) \leq C_3 N_m^3(t), \quad (3.189)$$

and

$$\begin{aligned} \|\chi F^\kappa\|_{\mathcal{H}^{1,\infty}}^2 &\leq C_4\{\|u\|_{\mathcal{H}^{1,\infty}}^2 \|\nabla u\|_{\mathcal{H}^{1,\infty}}^2 + \varepsilon^2(\|\nabla u\|_{\mathcal{H}^{3,\infty}}^2 + \|u\|_{\mathcal{H}^{3,\infty}}^2)\} \\ &\leq \delta\varepsilon \|\nabla^2 u\|_{\mathcal{H}^4}^2 + C_4\{\delta\varepsilon^3 N_m(t) + N_m^2(t)\}. \end{aligned} \quad (3.190)$$

It follows from (3.185)-(3.190) that

$$\|F\|_{\mathcal{H}^{1,\infty}}^2 \leq \delta\varepsilon \|\nabla^2 u\|_{\mathcal{H}^4}^2 + C_4\{C_\delta\varepsilon^3 N_m(t) + N_m^4(t)\}, \quad m \geq 6, \quad (3.191)$$

where  $F \triangleq \chi(\tilde{F}_1 \times n) + \chi \Pi(B\tilde{F}_2) + F^\chi + \chi F^\kappa$ . In order to be able to use Lemma 3.10, we shall perform last change of unknown in order to eliminate the term  $\partial_z(\ln|\tilde{g}|)\partial_z\tilde{\eta}$ . We set

$$\tilde{\eta} = \frac{1}{|g|^{\frac{1}{4}}}\bar{\eta} = \bar{\gamma} \bar{\eta}.$$

Note that we have

$$\|\tilde{\eta}\|_{\mathcal{H}^{1,\infty}} \leq C_3\|\bar{\eta}\|_{\mathcal{H}^{1,\infty}}, \quad \|\bar{\eta}\|_{\mathcal{H}^{1,\infty}} \leq C_3\|\tilde{\eta}\|_{\mathcal{H}^{1,\infty}} \quad (3.192)$$

and that,  $\bar{\eta}$  solves the equation

$$\begin{aligned} &\partial_t \bar{\eta} + \tilde{u}^1 \partial_{y_1} \bar{\eta} + \tilde{u}^2 \partial_{y_2} \bar{\eta} + \tilde{u} \cdot n \partial_z \bar{\eta} - \varepsilon \partial_{zz} \bar{\eta} \\ &= \frac{1}{\bar{\gamma}} \left( \tilde{F} + \varepsilon \partial_{zz} \bar{\gamma} \cdot \bar{\eta} + \frac{\varepsilon}{2} \partial_z(\ln|g|) \partial_z \bar{\gamma} \cdot \bar{\eta} - (\tilde{u} \cdot \nabla \bar{\gamma}) \bar{\eta} \right) := \mathcal{S}. \end{aligned}$$

Hence, it is easy to deduce for  $m \geq 6$

$$\begin{aligned} \|\mathcal{S}\|_{\mathcal{H}^{1,\infty}}^2 &\leq C_4(\|F\|_{\mathcal{H}^{1,\infty}}^2 + \|\bar{\eta}\|_{\mathcal{H}^{1,\infty}}^2 + \|\tilde{u}\|_{\mathcal{H}^{1,\infty}}^2 \|\bar{\eta}\|_{\mathcal{H}^{1,\infty}}^2) \\ &\leq C_4\{\delta\varepsilon \|\nabla^2 u\|_{\mathcal{H}^4}^2 + C_\delta\varepsilon^3 N_m(t) + N_m^4(t)\}. \end{aligned} \quad (3.193)$$

Consequently, by using Lemma 3.10, we get that for  $m \geq 6$

$$\begin{aligned} \|\bar{\eta}\|_{\mathcal{H}^{1,\infty}} &\lesssim \|\bar{\eta}_0\|_{\mathcal{H}^{1,\infty}} + \int_0^t ((\|\tilde{u}\|_{\mathcal{H}^{2,\infty}} + \|\partial_z \tilde{u}\|_{\mathcal{H}^{1,\infty}})(\|\bar{\eta}\|_{\mathcal{H}^{1,\infty}} + \|\bar{\eta}\|_{\mathcal{H}^{m_0+3}}) + \|\mathcal{S}\|_{\mathcal{H}^{1,\infty}}) d\tau \\ &\lesssim \|\bar{\eta}_0\|_{\mathcal{H}^{1,\infty}} + \int_0^t ((\|u\|_{\mathcal{H}^{2,\infty}} + \|\nabla u\|_{\mathcal{H}^{1,\infty}})(\|\tilde{\eta}\|_{\mathcal{H}^{1,\infty}} + \|\tilde{\eta}\|_{\mathcal{H}^{m_0+3}}) + \|\mathcal{S}\|_{\mathcal{H}^{1,\infty}}) d\tau \\ &\lesssim \|\bar{\eta}_0\|_{\mathcal{H}^{1,\infty}} + \int_0^t (N_m(\tau) + \|\mathcal{S}\|_{\mathcal{H}^{1,\infty}}) d\tau. \end{aligned} \quad (3.194)$$

Then, we deduce from (3.191)-(3.194) that

$$\|\eta(t)\|_{\mathcal{H}^{1,\infty}}^2 \leq \|\eta_0\|_{\mathcal{H}^{1,\infty}}^2 + CC_4 \left\{ \delta\varepsilon \int_0^t \|\nabla^2 u\|_{\mathcal{H}^4}^2 d\tau + \int_0^t N_m^4(\tau) d\tau \right\},$$

which, together with (3.177), gives directly

$$\|\nabla u\|_{\mathcal{H}^{1,\infty}}^2 \leq C_4 \left\{ N_m(0) + \|u\|_{\mathcal{H}^m}^2 + \delta\varepsilon \int_0^t \|\nabla^2 u\|_{\mathcal{H}^4}^2 d\tau + C_\delta \int_0^t N_m^4(\tau) d\tau \right\}.$$

Therefore, we complete the proof of the lemma 3.11.  $\square$

### 3.6. Proof of Theorem 3.1

By virtue of the definition of  $N_m(t)$  and  $Q(t)$ , one deduces from the estimates in Lemma 3.9 that

$$Q(t) \leq CC_4 P(N_m(t)),$$

which, together with (3.151), gives directly

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} (\| (u, \nabla d) \|_{\mathcal{H}^m}^2 + \| (\nabla u, \Delta d) \|_{\mathcal{H}^{m-1}}^2) + \varepsilon \int_0^t \|\nabla u\|_{\mathcal{H}^m}^2 d\tau \\ & + \varepsilon \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau + \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau + \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \\ & \leq CC_{m+2} \left\{ \|(u_0, \nabla d_0)\|_{\mathcal{H}^m}^2 + \|(\nabla u_0, \Delta d_0)\|_{\mathcal{H}^{m-1}}^2 + P(N_m(t)) \int_0^t N_m(t) d\tau \right\}. \end{aligned} \quad (3.195)$$

By virtue of the basic fact  $|d| = 1$  (see (3.7)), the combination of (3.176) and (3.195) yields that

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} N_m(\tau) + \varepsilon \int_0^t \|\nabla u\|_{\mathcal{H}^m}^2 d\tau + \varepsilon \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau + \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau \\ & + \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \leq \tilde{C}_2 C_{m+2} \left\{ N_m(0) + P(N_m(t)) \int_0^t P(N_m(\tau)) d\tau \right\}. \end{aligned}$$

Therefore, we complete the proof of Theorem 3.1.

## 4 Proof of Theorem 1.1 (Uniform Regularity)

In this section, we will give the proof for the Theorem 1.1. Indeed, we shall indicate how to combine the a priori estimates obtained so far to prove the uniform existence result. Fixing  $m \geq 6$ , we consider the initial data  $(u_0^\varepsilon, d_0^\varepsilon) \in X_{n,m}$  such that

$$\mathcal{I}_m(0) = \sup_{0 < \varepsilon \leq 1} \|(u_0^\varepsilon, d_0^\varepsilon)\|_{X_{n,m}} \leq \tilde{C}_0. \quad (4.1)$$

For such initial data, since we are not aware of a local existence result for (1.1) and (1.2) (or (1.3)), we first establish the local existence of solution for (1.1) and (1.2) with initial data  $(u_0^\varepsilon, d_0^\varepsilon) \in X_{n,m}$ . For such initial data  $(u_0^\varepsilon, d_0^\varepsilon)$ , it is easy to see that there exists a sequence of smooth approximate initial data  $(u_0^{\varepsilon,\delta}, d_0^{\varepsilon,\delta}) \in X_{n,m}^{ap}$  ( $\delta$  being a regularity parameter), which has enough space regularity so that the time derivatives at the initial time can be defined by the equations (1.1) and the boundary compatibility conditions are satisfied. Fixed  $\varepsilon \in (0, 1]$ , one constructs the approximate solutions as follows:

(1) Define  $u^0 = u_0^{\varepsilon,\delta}$  and  $d^0 = d_0^{\varepsilon,\delta}$ .

(2) Assume that  $(u^{k-1}, d^{k-1})$  has been defined for  $k \geq 1$ . Let  $(u^k, d^k)$  be the unique solution to the following linearized initial data boundary value problem:

$$\begin{cases} u_t^k + u^{k-1} \cdot \nabla u^k - \varepsilon \Delta u^k + \nabla p^k = -\nabla d^k \cdot \Delta d^k, & (x, t) \in \Omega \times (0, T), \\ d_t^k - \Delta d^k = |\nabla d^{k-1}|^2 d^{k-1} - u^{k-1} \cdot \nabla d^{k-1}, & (x, t) \in \Omega \times (0, T), \\ \operatorname{div} u^k = 0, & (x, t) \in \Omega \times (0, T), \end{cases} \quad (4.2)$$

with initial data

$$(u^k, d^k)|_{t=0} = (u_0^{\varepsilon, \delta}, d_0^{\varepsilon, \delta}), \quad (4.3)$$

and Navier-type and Neumann boundary condition

$$u^k \cdot n = 0, \quad n \times (\nabla \times u^k) = [Bu^k]_\tau, \quad \text{and} \quad \frac{\partial d^k}{\partial n} = 0, \quad \text{on } \partial\Omega. \quad (4.4)$$

Since  $u^k$  and  $d^k$  are decoupled, the existence of global unique smooth solution  $(u^k, d^k)(t)$  of (4.2)-(4.4) can be obtained by using classical methods, for example, refer to [23] or [24]. On the other hand, by virtue of  $(u_0^{\varepsilon, \delta}, d_0^{\varepsilon, \delta}) \in H^{4m} \times H^{4(m+1)}$ , one proves that there exists a positive time  $\tilde{T}_1 = \tilde{T}_1(\varepsilon)$  (depending on  $\varepsilon$ ,  $\|u_0^{\varepsilon, \delta}\|_{H^{4m}}$  and  $\|d_0^{\varepsilon, \delta}\|_{H^{4(m+1)}}$ ) such that

$$\|u^k(t)\|_{H^{4m}}^2 + \|d^k(t)\|_{H^{4(m+1)}}^2 \leq \tilde{C}_1 \quad \text{for } 0 \leq t \leq \tilde{T}_1, \quad (4.5)$$

where the constant  $\tilde{C}_1$  depends on  $\tilde{C}_0, \varepsilon^{-1}, \|u_0^{\varepsilon, \delta}\|_{H^{4m}}$  and  $\|d_0^{\varepsilon, \delta}\|_{H^{4(m+1)}}$ . Based on the above uniform time  $\hat{T}_1(\leq \tilde{T}_1)$  (independent of  $k$ ) such that  $(u^k, d^k)$  converges to a limit  $(u^{\varepsilon, \delta}, d^{\varepsilon, \delta})$  as  $k \rightarrow +\infty$  in the following strong sense:

$$u^k \rightarrow u^{\varepsilon, \delta} \text{ in } L^\infty(0, \hat{T}_1; L^2) \quad \text{and} \quad \nabla u^k \rightarrow \nabla u^{\varepsilon, \delta} \text{ in } L^2(0, \hat{T}_1; L^2),$$

and

$$d^k \rightarrow d^{\varepsilon, \delta} \text{ in } L^\infty(0, \hat{T}_1; H^1) \quad \text{and} \quad \Delta d^k \rightarrow \Delta d^{\varepsilon, \delta} \text{ in } L^2(0, \hat{T}_1; L^2).$$

It is easy to check that  $(u^{\varepsilon, \delta}, d^{\varepsilon, \delta})$  is a classical solution to the problem (1.1) and (1.2) with initial data  $(u_0^{\varepsilon, \delta}, d_0^{\varepsilon, \delta})$ . In view of the lower semicontinuity of norms, one can deduce from the uniform bounds (4.5) that

$$\|u^{\varepsilon, \delta}(t)\|_{H^{4m}}^2 + \|d^{\varepsilon, \delta}(t)\|_{H^{4(m+1)}}^2 \leq \tilde{C}_1 \quad \text{for } 0 \leq t \leq \tilde{T}_1. \quad (4.6)$$

Applying the a priori estimates given in Theorem 3.1 to the solution  $(u^{\varepsilon, \delta}, d^{\varepsilon, \delta})$ , one can obtain a uniform time  $T_0$  and constant  $C_3$  (independent of  $\varepsilon$  and  $\delta$ ) such that for all  $t \in [0, \min\{T_0, \hat{T}_1\}]$

$$\sup_{0 \leq \tau \leq t} N_m(\tau) + \varepsilon \int_0^t (\|\nabla u\|_{\mathcal{H}^m}^2 + \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2) d\tau + \int_0^t (\|\Delta d\|_{\mathcal{H}^m}^2 + \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2) d\tau \leq \tilde{C}_3, \quad (4.7)$$

where  $T_0$  and  $\tilde{C}_3$  depend only on  $\hat{C}_0$  and  $\mathcal{I}_m(0)$ . Based on the uniform estimate (4.7) for  $(u^{\varepsilon, \delta}, d^{\varepsilon, \delta})$ , one can pass the limit  $\delta \rightarrow 0$  to get a strong solution  $(u^\varepsilon, d^\varepsilon)$  of (1.1) and (1.2) with initial data  $(u_0^\varepsilon, d_0^\varepsilon)$  satisfying (4.2) by using a strong compactness arguments (see [25]). Indeed, it follows from (4.7) that  $(u^{\varepsilon, \delta}, \nabla d^{\varepsilon, \delta})$  is bounded uniformly in  $L^\infty(0, \tilde{T}_2; H_{co}^m)$ , where  $\tilde{T}_2 = \min\{T_0, \tilde{T}_1\}$ , while  $(\nabla u^{\varepsilon, \delta}, \Delta d^{\varepsilon, \delta})$  is bounded uniformly in  $L^\infty(0, \tilde{T}_2; H_{co}^{m-1})$ , and  $(\partial_t u^{\varepsilon, \delta}, \partial_t \nabla d^{\varepsilon, \delta})$  is bounded uniformly in  $L^\infty(0, \tilde{T}_2; H_{co}^{m-1})$ . Then, the strong compactness argument implies that  $(u^{\varepsilon, \delta}, \nabla d^{\varepsilon, \delta})$  is compact in  $\mathcal{C}([0, \tilde{T}_2]; H_{co}^{m-1})$ . In particular, there exists a sequence  $\delta_n \rightarrow 0^+$  and  $(u^\varepsilon, \nabla d^\varepsilon) \in \mathcal{C}([0, \tilde{T}_2]; H_{co}^{m-1})$  such that

$$(u^{\varepsilon, \delta_n}, \nabla d^{\varepsilon, \delta_n}) \rightarrow (u^\varepsilon, \nabla d^\varepsilon) \text{ in } \mathcal{C}([0, \tilde{T}_2]; H_{co}^{m-1}) \text{ as } \delta_n \rightarrow 0^+.$$

Moreover, applying the lower semicontinuity of norms to the bounds (4.7), one obtains the bounds (4.7) for  $(u^\varepsilon, d^\varepsilon)$ . It follows from the bounds of (4.7) for  $(u^\varepsilon, d^\varepsilon)$ , and the anisotropic Sobolev inequality (2.5) that

$$\begin{aligned} & \sup_{0 \leq t \leq \tilde{T}_2} \|(u^{\varepsilon, \delta_n} - u^\varepsilon, d^{\varepsilon, \delta_n} - d^\varepsilon)\|_{L^\infty}^2 \\ & \leq C \sup_{0 \leq t \leq \tilde{T}_2} \|\nabla(u^{\varepsilon, \delta_n} - u^\varepsilon, d^{\varepsilon, \delta_n} - d^\varepsilon)\|_{H_{co}^1} \|(u^{\varepsilon, \delta_n} - u^\varepsilon, d^{\varepsilon, \delta_n} - d^\varepsilon)\|_{H_{co}^2} \rightarrow 0, \end{aligned}$$



and

$$\sup_{0 \leq t \leq \tilde{T}_2} \|\nabla(d^{\varepsilon, \delta_n} - d^\varepsilon)\|_{L^\infty}^2 \leq C \sup_{0 \leq t \leq \tilde{T}_2} \|\Delta(d^{\varepsilon, \delta_n} - d^\varepsilon)\|_{H_{co}^1} \|\nabla(d^{\varepsilon, \delta_n} - d^\varepsilon)\|_{H_{co}^2} \rightarrow 0,$$

Hence, it is easy to check that  $(u^\varepsilon, d^\varepsilon)$  is a weak solution of the nematic liquid crystal flows (1.1). The uniqueness of the solution  $(u^\varepsilon, d^\varepsilon)$  comes directly from the Lipschitz regularity of solution. Thus, the whole family  $(u^{\varepsilon, \delta}, d^{\varepsilon, \delta})$  converge to  $(u^\varepsilon, d^\varepsilon)$ . Therefore, we have established the local solution of equation (1.1) and (1.2) with initial data  $(u_0^\varepsilon, d_0^\varepsilon) \in X_{n,m}$ ,  $t \in [0, T_2]$ .

We shall use the local existence results to prove Theorem 1.1. If  $T_0 \leq \tilde{T}$ , then Theorem 1.1 follows from (4.7) with  $\tilde{C}_1 = \tilde{C}_3$ . On the other hand, for the case  $\tilde{T} \leq T_0$ , based on the uniform estimate (4.7), we can use the local existence results established above to extend our solution step by step to the uniform time interval  $t \in [0, T_0]$ . Therefore, we complete the proof of Theorem 1.1.

## 5 Proof of Theorem 1.2 (Inviscid Limit)

In this section, we study the vanishing viscosity of solutions for the equation (1.1) to the solution for the equation (1.4) with a rate of convergence. It is easy to see that the solution  $(u, d) \in H^3 \times H^4$  of equation (1.1) and (1.2) with initial data  $(u_0, d_0) \in H^3 \times H^4$  satisfies

$$\|u\|_{C([0, T_1]; H^3)} + \|d\|_{C([0, T_1]; H^4)} \leq \tilde{C}_4$$

where  $\tilde{C}_4$  depends only on  $\|(u_0, d_0)\|_{H^3 \times H^4}$ . On the other hand, it follows from the Theorem 1.1 that the solution  $(u^\varepsilon, d^\varepsilon)$  of equation (1.1) and (1.2) with initial data  $(u_0, d_0)$  satisfies

$$\|(u^\varepsilon, d^\varepsilon)\|_{X_m} \leq \tilde{C}_1, \quad \forall t \in [0, T_0],$$

where  $T_0$  and  $\tilde{C}_1$  are defined in Theorem 1.1. In particular, this uniform regularity implies the bound

$$\|u^\varepsilon\|_{W^{1, \infty}} + \|d^\varepsilon\|_{W^{2, \infty}} \leq \tilde{C}_1,$$

which plays an important role in the proof of Theorem 1.2.

Let us define

$$v^\varepsilon = u^\varepsilon - u, \quad \varphi^\varepsilon = d^\varepsilon - d.$$

It then follows from (1.1) that

$$\begin{cases} \partial_t v^\varepsilon + u \cdot \nabla v^\varepsilon + \varepsilon \nabla \times (\nabla \times v^\varepsilon) + \nabla(p^\varepsilon - p) = R_1^\varepsilon, \\ \operatorname{div} v^\varepsilon = 0, \\ \partial_t \varphi^\varepsilon + u \cdot \nabla \varphi^\varepsilon - \Delta \varphi^\varepsilon = R_2^\varepsilon, \end{cases} \quad (5.1)$$

where

$$\begin{aligned} R_1^\varepsilon &\triangleq \varepsilon \Delta u - v^\varepsilon \cdot \nabla u^\varepsilon - \nabla d^\varepsilon \cdot \Delta \varphi^\varepsilon - \nabla \varphi^\varepsilon \cdot \Delta d, \\ R_2^\varepsilon &\triangleq -v^\varepsilon \cdot \nabla d^\varepsilon + (\nabla \varphi^\varepsilon : \nabla(d^\varepsilon + d))d^\varepsilon + |\nabla d|^2 \varphi^\varepsilon. \end{aligned}$$

The boundary conditions to (5.1) are

$$\begin{cases} v^\varepsilon \cdot n = 0, & n \times (\nabla \times v^\varepsilon) = [Bv^\varepsilon]_\tau + [Bu]_\tau - n \times w, \quad x \in \partial\Omega, \\ \frac{\partial \varphi^\varepsilon}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \quad (5.2)$$

**Lemma 5.1.** For  $t \in [0, \min\{T_0, T_1\}]$ , it holds that

$$\sup_{0 \leq \tau \leq t} (\|v^\varepsilon(\tau)\|_{L^2} + \|\varphi^\varepsilon(\tau)\|_{H^1}^2) + \varepsilon \int_0^t \int |\nabla v^\varepsilon|^2 dx d\tau + \int_0^t \int (|\nabla \varphi^\varepsilon|^2 + |\Delta \varphi^\varepsilon|^2) dx d\tau \leq C\varepsilon^{\frac{3}{2}}. \quad (5.3)$$

where  $C > 0$  depend only on  $\tilde{C}_0, \tilde{C}_1$  and  $\tilde{C}_4$ .

*Proof.* Multiplying (5.1)<sub>1</sub> by  $v^\varepsilon$  and integrating by parts, we find

$$\frac{1}{2} \frac{d}{dt} \int |v^\varepsilon|^2 dx + \varepsilon \int \nabla \times (\nabla \times v^\varepsilon) \cdot v^\varepsilon dx = \int R_1^\varepsilon \cdot v^\varepsilon dx. \quad (5.4)$$

Integrating by part and applying the boundary condition (5.2), one arrives at directly

$$\begin{aligned} & \int \nabla \times (\nabla \times v^\varepsilon) \cdot v^\varepsilon dx \\ &= \int_{\partial\Omega} n \times (\nabla \times v^\varepsilon) \cdot v^\varepsilon dx + \int |\nabla \times v^\varepsilon|^2 dx \\ &= \int_{\partial\Omega} ([Bv^\varepsilon]_\tau + [Bu]_\tau - n \times w) \cdot v^\varepsilon d\sigma + \int |\nabla \times v^\varepsilon|^2 dx \\ &\leq C|v^\varepsilon|_{L^2(\partial\Omega)}^2 + C|v^\varepsilon|_{L^2(\partial\Omega)} + \int |\nabla \times v^\varepsilon|^2 dx. \end{aligned} \quad (5.5)$$

The application of Proposition 2.1 gives directly

$$\|\nabla v^\varepsilon\|_{L^2}^2 \leq C(\|\nabla \times v^\varepsilon\|_{L^2}^2 + \|v^\varepsilon\|_{L^2}^2). \quad (5.6)$$

The combination of (5.4)-(5.6) yields that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |v^\varepsilon|^2 dx + C_1 \varepsilon \int |\nabla v^\varepsilon|^2 dx \\ & \leq \int R_1^\varepsilon \cdot v^\varepsilon dx + C\varepsilon|v^\varepsilon|_{L^2(\partial\Omega)}^2 + C\varepsilon|v^\varepsilon|_{L^2(\partial\Omega)} + C\varepsilon\|v^\varepsilon\|_{L^2}^2. \end{aligned} \quad (5.7)$$

By virtue of the Hölder and Cauchy inequalities, we obtain

$$\begin{aligned} \int R_1^\varepsilon \cdot v^\varepsilon dx &\leq \delta \|\Delta \varphi^\varepsilon\|_{L^2}^2 + \varepsilon \|\Delta u\|_{L^\infty} \|v^\varepsilon\|_{L^2} + \|\nabla u^\varepsilon\|_{L^\infty} \|v^\varepsilon\|_{L^2}^2 \\ &\quad + C_\delta \|\nabla d^\varepsilon\|_{L^\infty}^2 \|v^\varepsilon\|_{L^2}^2 + \|\Delta d\|_{L^\infty} \|\nabla \varphi^\varepsilon\|_{L^2} \|v^\varepsilon\|_{L^2} \\ &\leq \delta \|\Delta \varphi^\varepsilon\|_{L^2}^2 + \varepsilon \|v^\varepsilon\|_{L^2} + C_\delta (\|\nabla \varphi^\varepsilon\|_{L^2}^2 + \|v^\varepsilon\|_{L^2}^2), \end{aligned}$$

which, together with (5.7), gives immediately

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int |v^\varepsilon|^2 dx + C_1 \varepsilon \int |\nabla v^\varepsilon|^2 dx \\ & \leq C\varepsilon|v^\varepsilon|_{L^2(\partial\Omega)}^2 + C\varepsilon|v^\varepsilon|_{L^2(\partial\Omega)} + \delta \|\Delta \varphi^\varepsilon\|_{L^2}^2 + \varepsilon \|v^\varepsilon\|_{L^2} + C_\delta (\|\nabla \varphi^\varepsilon\|_{L^2}^2 + \|v^\varepsilon\|_{L^2}^2). \end{aligned} \quad (5.8)$$

Multiplying (5.1) by  $-\Delta \varphi^\varepsilon$  and integrating over  $\Omega$ , we find

$$-\int \partial_t \varphi^\varepsilon \cdot \Delta \varphi^\varepsilon dx + \int |\Delta \varphi^\varepsilon|^2 dx = -\int (u \cdot \nabla \varphi^\varepsilon) \cdot \Delta \varphi^\varepsilon dx - \int R_2^\varepsilon \cdot \Delta \varphi^\varepsilon dx. \quad (5.9)$$

Integrating by part and applying the boundary condition (5.2), it holds that

$$-\int \partial_t \varphi^\varepsilon \cdot \Delta \varphi^\varepsilon dx = -\int_{\partial\Omega} \partial_t \varphi^\varepsilon \cdot (n \cdot \nabla \varphi^\varepsilon) d\sigma + \frac{1}{2} \frac{d}{dt} \int |\nabla \varphi^\varepsilon|^2 dx = \frac{1}{2} \frac{d}{dt} \int |\nabla \varphi^\varepsilon|^2 dx. \quad (5.10)$$

Applying the Cauchy inequality, it is easy to deduce that

$$\begin{aligned} & -\int (u \cdot \nabla \varphi^\varepsilon) \cdot \Delta \varphi^\varepsilon dx - \int R_2^\varepsilon \cdot \Delta \varphi^\varepsilon dx \\ & \leq \delta \|\Delta \varphi^\varepsilon\|_{L^2}^2 + C_\delta \|u\|_{L^\infty}^2 \|\nabla \varphi^\varepsilon\|_{L^2}^2 + C_\delta \|\nabla d^\varepsilon\|_{L^\infty}^2 (\|\nabla \varphi^\varepsilon\|_{L^2}^2 + \|v^\varepsilon\|_{L^2}^2) \\ & \quad + C_\delta \|\nabla d\|_{L^\infty}^2 (\|\nabla \varphi^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{L^2}^2) \\ & \leq \delta \|\Delta \varphi^\varepsilon\|_{L^2}^2 + C_\delta (\|v^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{L^2}^2 + \|\nabla \varphi^\varepsilon\|_{L^2}^2). \end{aligned} \quad (5.11)$$

Substituting (5.10)-(5.11) into (5.9), we obtain

$$\frac{1}{2} \frac{d}{dt} \int |\nabla \varphi^\varepsilon|^2 dx + \frac{3}{4} \int |\Delta \varphi^\varepsilon|^2 dx \leq C(\|v^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{L^2}^2 + \|\nabla \varphi^\varepsilon\|_{L^2}^2). \quad (5.12)$$

In order to control the term  $\int |\varphi^\varepsilon|^2 dx$  on the right hand side of (5.12), we multiply the equation (5.1)<sub>3</sub> by  $\varphi^\varepsilon$  and integrating by part to get that

$$\frac{1}{2} \frac{d}{dt} \int |\varphi^\varepsilon|^2 dx + \int |\nabla \varphi^\varepsilon|^2 dx = \int R_2^\varepsilon \cdot \varphi^\varepsilon dx. \quad (5.13)$$

In view of the Hölder inequality, one arrives at

$$\begin{aligned} \int R_2^\varepsilon \cdot \varphi^\varepsilon dx & \leq \|\nabla d^\varepsilon\|_{L^\infty} (\|v^\varepsilon\|_{L^2} \|\varphi^\varepsilon\|_{L^2} + \|\varphi^\varepsilon\|_{L^2} \|\nabla \varphi^\varepsilon\|_{L^2}) \\ & \quad + \|\nabla d\|_{L^\infty} \|\nabla \varphi^\varepsilon\|_{L^2} \|\varphi^\varepsilon\|_{L^2} + \|\nabla d\|_{L^\infty}^2 \|\varphi^\varepsilon\|_{L^2}^2 \\ & \leq C(\|v^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{L^2}^2 + \|\nabla \varphi^\varepsilon\|_{L^2}^2), \end{aligned}$$

which, together with (5.8), (5.12) and (5.13), yields directly

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int (|v^\varepsilon|^2 + |\varphi^\varepsilon|^2 + |\nabla \varphi^\varepsilon|^2) dx + C_1 \varepsilon \int |\nabla v^\varepsilon|^2 dx + \frac{3}{4} \int (|\nabla \varphi^\varepsilon|^2 + |\Delta \varphi^\varepsilon|^2) dx \\ & \leq C\varepsilon \|v^\varepsilon\|_{L^2(\partial\Omega)}^2 + C\varepsilon \|v^\varepsilon\|_{L^2(\partial\Omega)} + \varepsilon^2 + C(\|v^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{L^2}^2 + \|\nabla \varphi^\varepsilon\|_{L^2}^2). \end{aligned} \quad (5.14)$$

By virtue of the trace theorem in Proposition 2.3 and Cauchy inequality, one deduces that

$$\|v^\varepsilon\|_{L^2(\partial\Omega)}^2 \leq \delta \|\nabla v^\varepsilon\|_{L^2}^2 + C_\delta \|v^\varepsilon\|_{L^2}^2, \quad (5.15)$$

and

$$\begin{aligned} \varepsilon \|v^\varepsilon\|_{L^2(\partial\Omega)} & \leq \varepsilon \|v^\varepsilon\|_{L^2}^{\frac{1}{2}} \|\nabla v^\varepsilon\|_{L^2}^{\frac{1}{2}} \leq \delta \varepsilon \|\nabla v^\varepsilon\|_{L^2}^2 + C_\delta \varepsilon \|\nabla v^\varepsilon\|_{L^2}^{\frac{3}{2}} \\ & \leq \delta \varepsilon \|\nabla v^\varepsilon\|_{L^2}^2 + C_\delta \|\nabla v^\varepsilon\|_{L^2}^2 + \varepsilon^{\frac{3}{2}}. \end{aligned} \quad (5.16)$$

Then the combination of (5.14)-(5.16) yields immediately

$$\begin{aligned} & \frac{d}{dt} \int (|v^\varepsilon|^2 + |\varphi^\varepsilon|^2 + |\nabla \varphi^\varepsilon|^2) dx + C_1 \varepsilon \int |\nabla v^\varepsilon|^2 dx + \int (|\nabla \varphi^\varepsilon|^2 + |\Delta \varphi^\varepsilon|^2) dx \\ & \leq \varepsilon^{\frac{3}{2}} + C(\|v^\varepsilon\|_{L^2} + \|\varphi^\varepsilon\|_{L^2}^2 + \|\nabla \varphi^\varepsilon\|_{L^2}^2), \end{aligned}$$

which, together with the Grönwall inequality, completes the proof of the lemma 5.1.  $\square$

**Remark 5.1.** By virtue of the nonlinear terms  $\nabla d^\varepsilon \cdot \Delta d^\varepsilon$  and  $\nabla d \cdot \Delta d$  on the right hand side of (1.1)<sub>1</sub> and (1.4)<sub>1</sub> respectively, we can not obtain the convergence rate for the quantity  $\|\nabla(u - u^\varepsilon)\|$  or  $\|\nabla^2(d - d^\varepsilon)\|$  in this paper.

**Proof for Theorem 1.2:** Indeed, the estimate (1.20) just follows from the estimate (5.3) in Lemma 5.1. On the other hand, by virtue of Sobolev inequality, uniform estimate (1.17) and convergence rate (5.3), it is easy to deduce

$$\|u^\varepsilon - u\|_{L^\infty(0,T_2;L^\infty(\Omega))} \leq C \|u^\varepsilon - u\|_{L^2}^{\frac{2}{5}} \|u^\varepsilon - u\|_{W^{1,\infty}}^{\frac{3}{5}} \leq C \varepsilon^{\frac{3}{10}},$$

and

$$\|d^\varepsilon - d\|_{L^\infty(0,T^2;W^{1,\infty}(\Omega))} \leq C \|d^\varepsilon - d\|_{H^1}^{\frac{2}{5}} \|d^\varepsilon - d\|_{W^{2,\infty}}^{\frac{3}{5}} \leq C \varepsilon^{\frac{3}{10}},$$

which implies the convergence rate (1.21). Therefore, we complete the proof of Theorem 1.2.

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